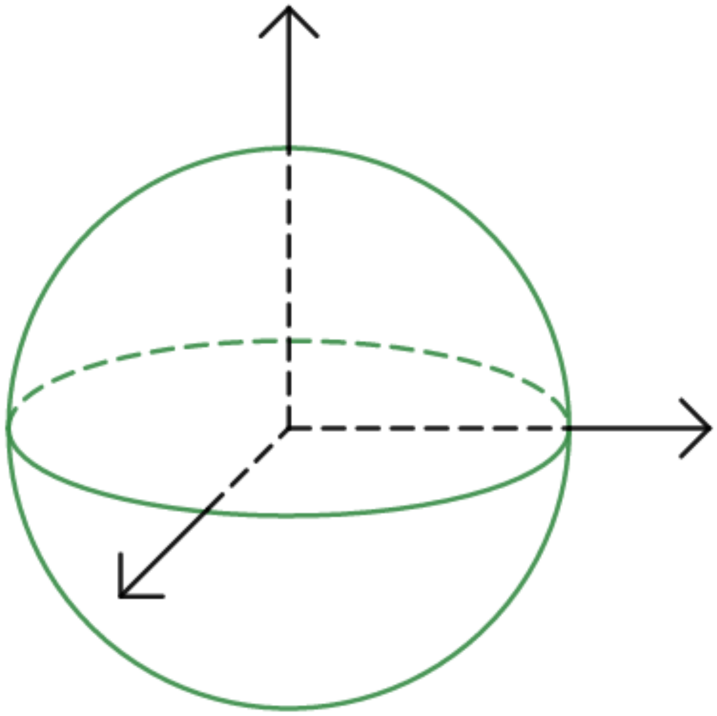
Chapter 15

Multiple Integrals

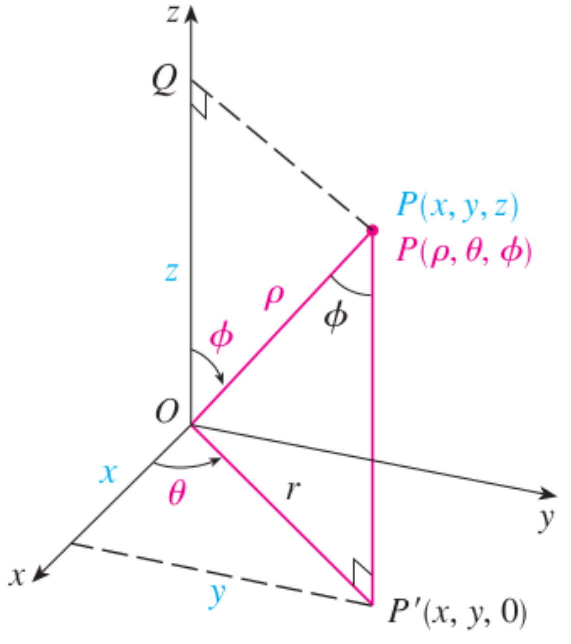
15.8 Triple integrals in spherical coordinates

Spherical coordinates

EXAMPLE. Describe the solid bounded by the sphere (picture below).



Definition



Cartesian $\longrightarrow$ Spherical

$$x = \rho \sin\phi \cos\theta$$
$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$

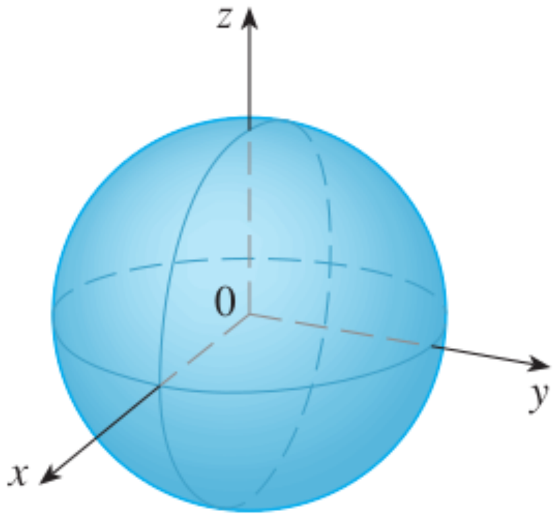
EXAMPLE 1 The point $(2, \pi/4, \pi/3)$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

EXAMPLE 2 The point $\left(0, 2\sqrt{3}, -2\right)$ is given in rectangular coordinates. Find spherical coordinates for this point.

Equations of important solids.

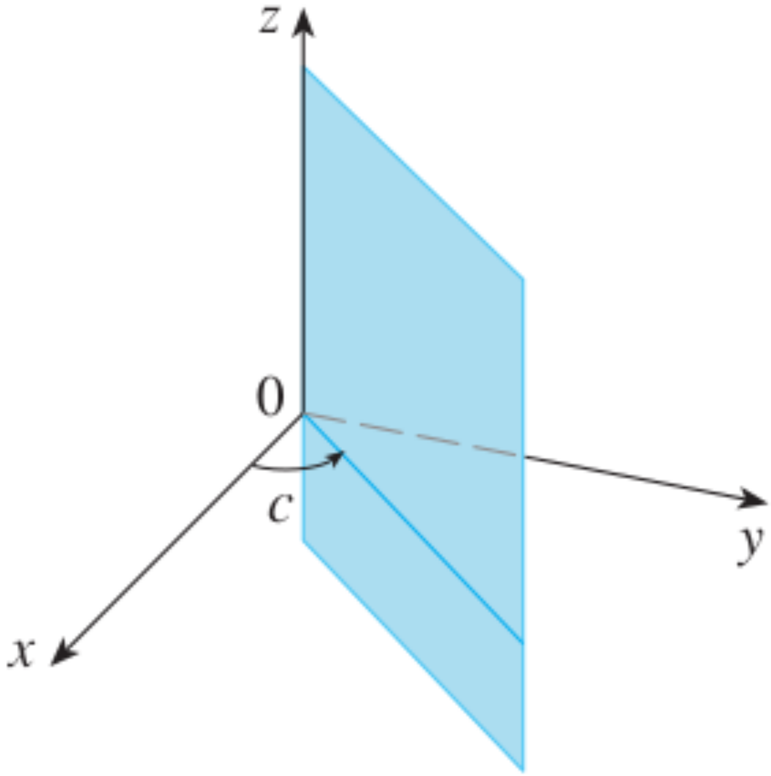
Sphere of radius R.

$$\boxed{\rho = R}$$



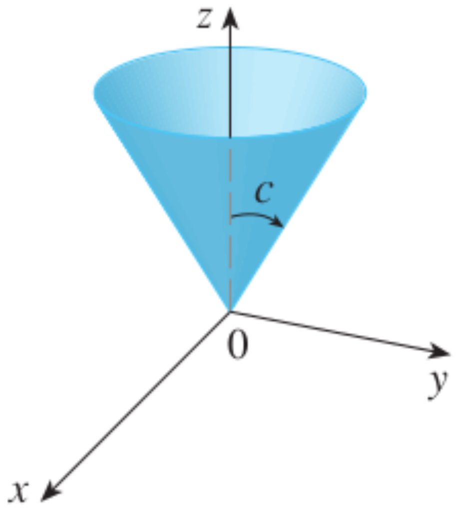
Half planes.

$$\boxed{\theta = c}$$

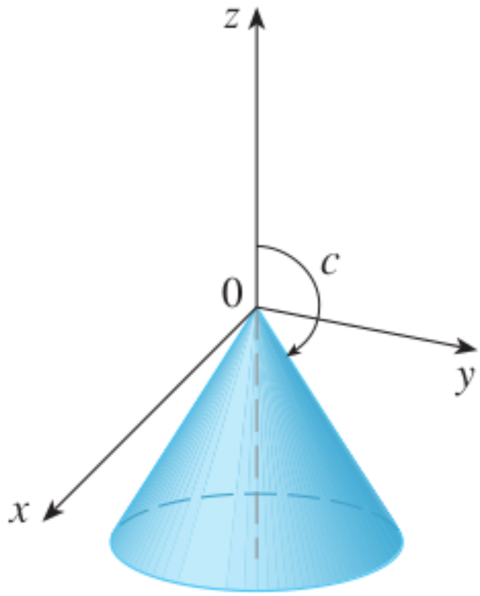


Cones.

$$\boxed{\phi = c}$$

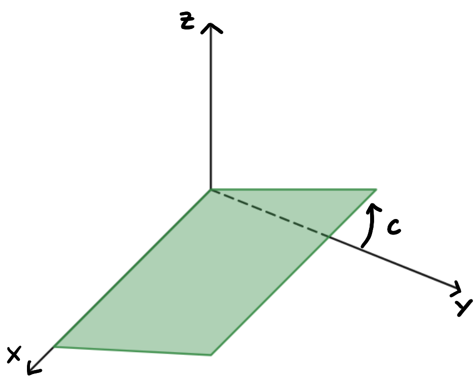


$0 < c < \pi/2$



$\pi/2 < c < \pi$

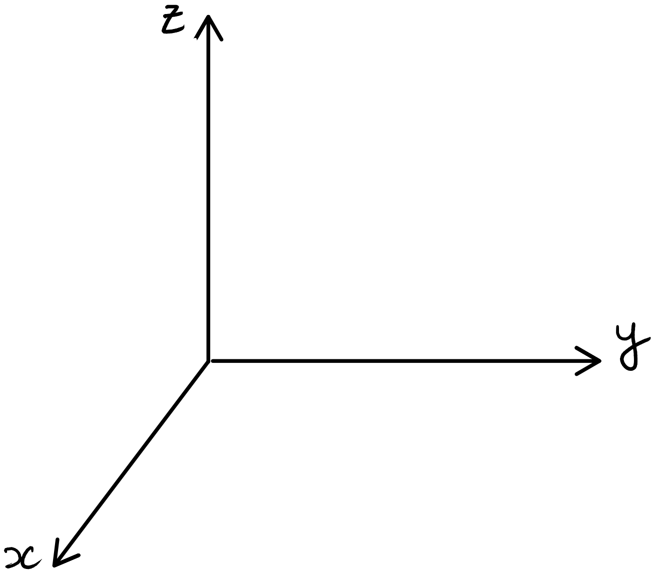
Question. Find the equation of the half-plane in the picture below in spherical coordinates. The plane is making an angle of c with the xy-plane.

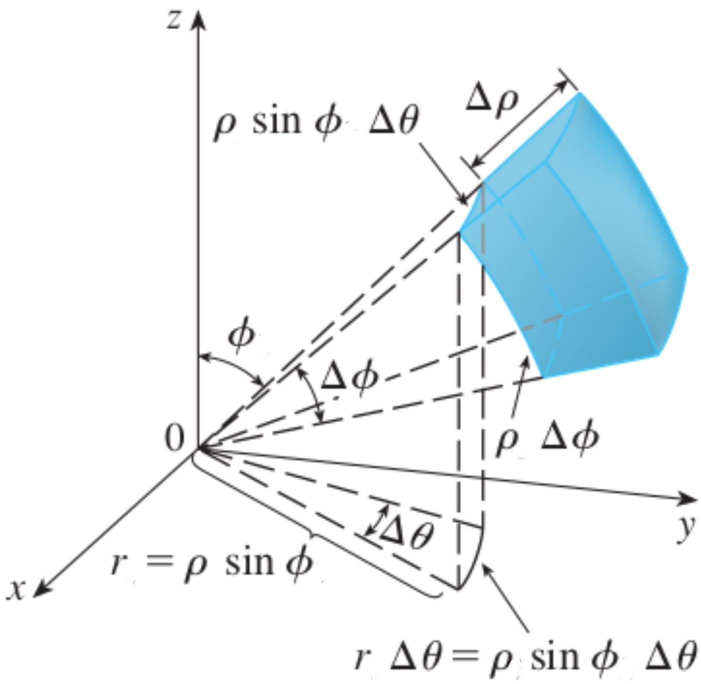


Evaluating integrals in sperical coordinates.

Spherical Wedge

$$E = \{(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho\cos\phi) \,:\, a \le \rho \le b,\, \alpha \le \theta \le \beta,\, c \le \phi \le d\}$$





We can show that

$$\Delta V = \rho^2 \sin\phi\, \Delta\rho\, \Delta\theta\, \Delta\phi$$

As the number of subdivisions goes to infinity, we obtain

$$\boxed{dV = \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi}$$

Formula for the change of variable (in spherical coordinates).

$$\boxed{\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\ \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi}$$

EXAMPLE 3 Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV$, where B is the unit ball:

$$B = \{(x, y, z) \mid x^2 + y^2 + z^2 \leqslant 1\}$$

EXAMPLE 4 Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.