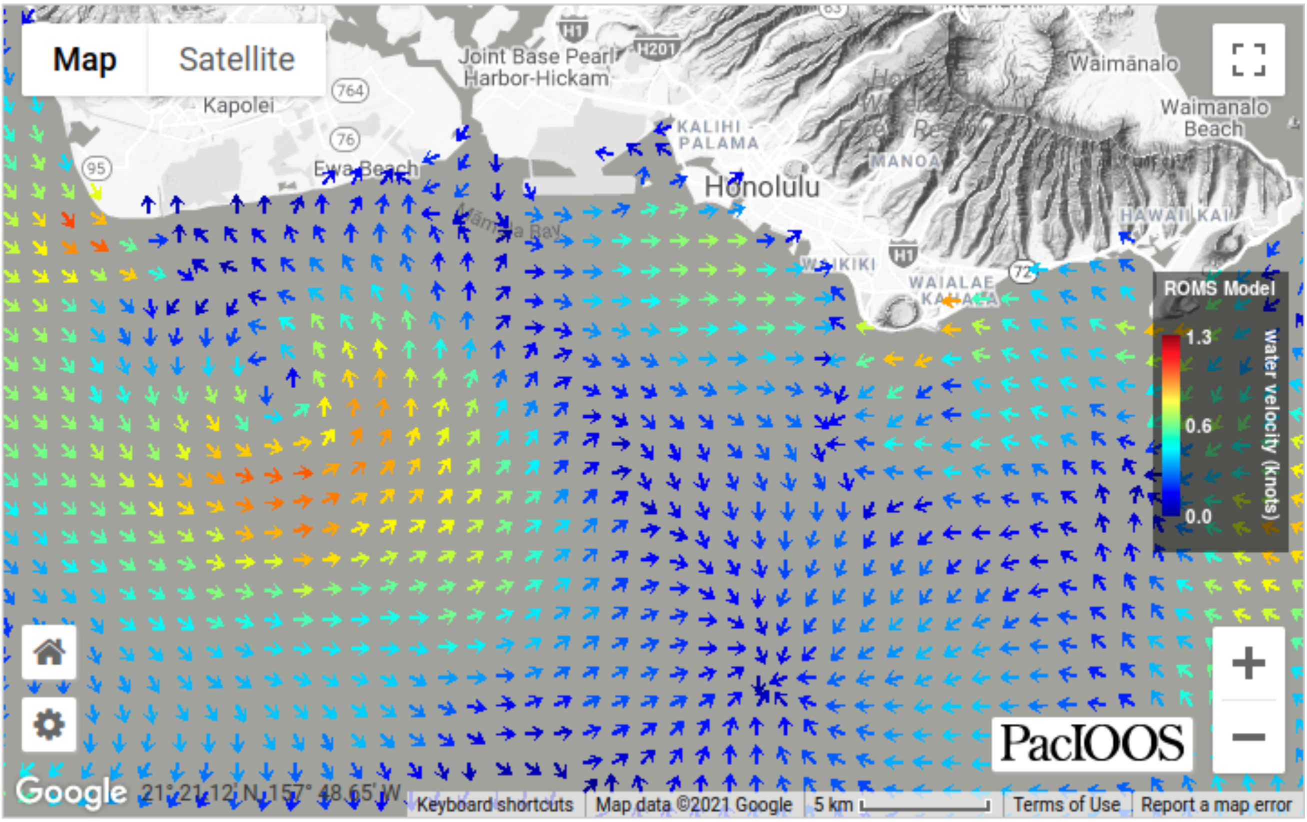
# Chapter 16

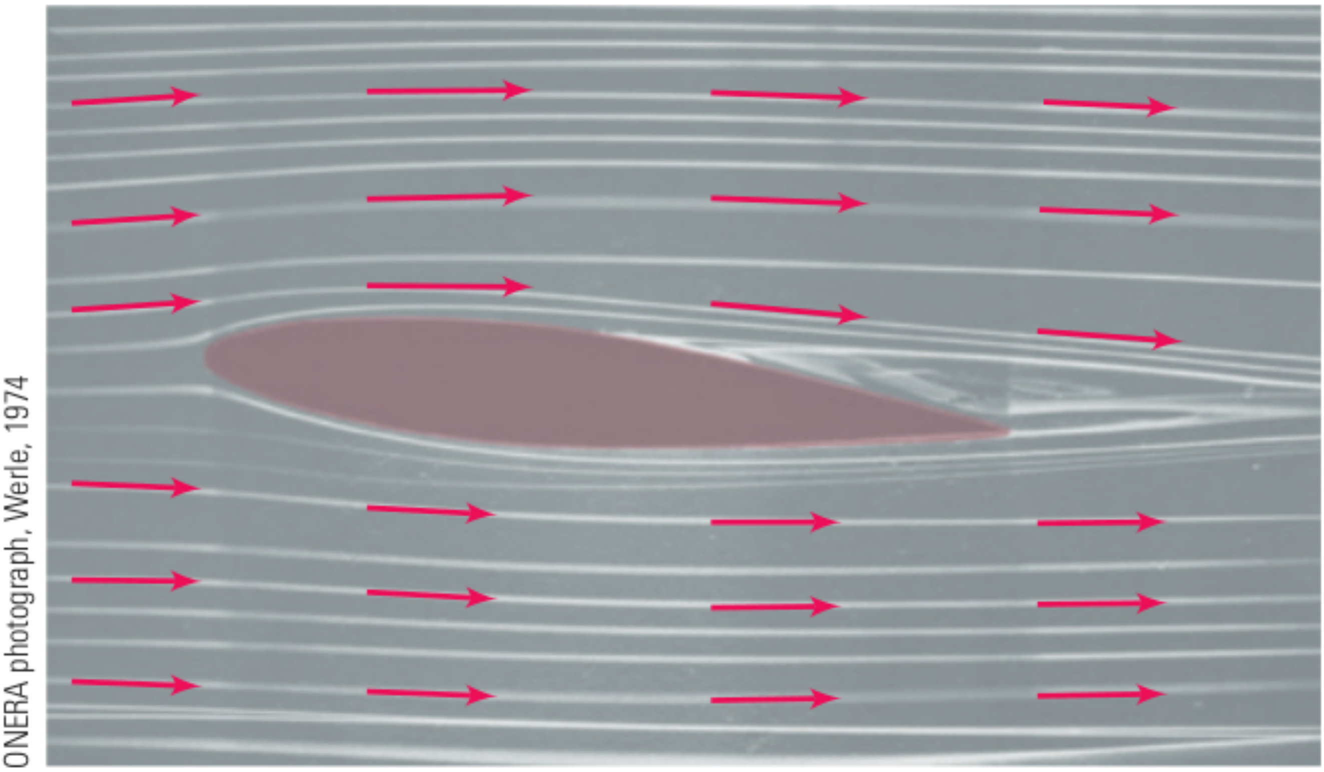
# Vector Calculus

## 16.1 Vector Fields

## Examples.




Map retrieved from http://www.pacioos.hawaii.edu/currents/model-oahu/

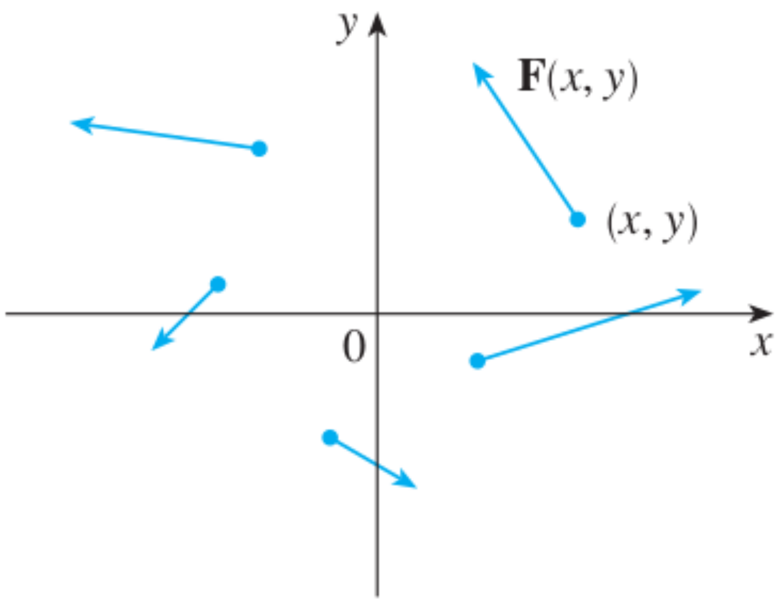



(b) Airflow past an inclined airfoil

## Vector Fields in 2D.

**1 Definition** Let $D$ be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point $(x, y)$ in $D$ a two-dimensional vector $\mathbf{F}(x, y)$.

Representation.




Component Functions

$$\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$$

- $P$ : $x$-component of $\vec{F}$
- $Q$ : $y$-component of $\vec{F}$

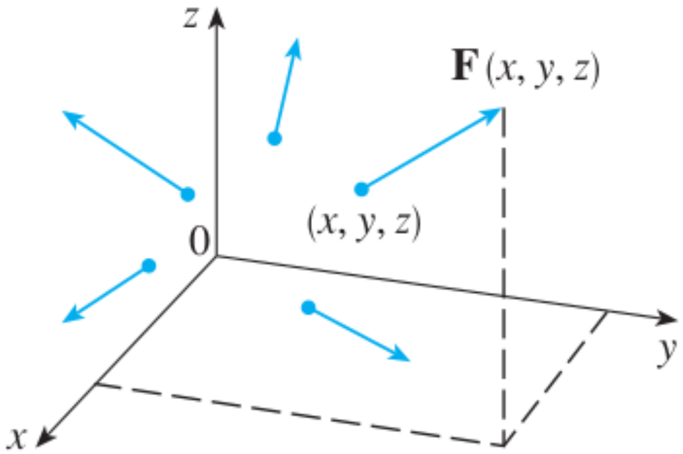
Remark:

**EXAMPLE 1** A vector field on $\mathbb{R}^2$ is defined by $\mathbf{F}(x, y) = -y\,\mathbf{i} + x\,\mathbf{j}$. Describe $\mathbf{F}$ by sketching some of the vectors $\mathbf{F}(x, y)$ as in Figure 3.

# Vector Fields in 3D.

**2 Definition** Let $E$ be a subset of $\mathbb{R}^3$. A **vector field on** $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point $(x, y, z)$ in $E$ a three-dimensional vector $\mathbf{F}(x, y, z)$.

Representation.




Component Functions.

$$P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$$

- $P$ : $x$-component of $\vec{F}$
- $Q$ : $y$-component of $\vec{F}$
- $R$ : $z$-component of $\vec{F}$

**EXAMPLE 2** Sketch the vector field on $\mathbb{R}^3$ given by $\mathbf{F}(x, y, z) = z\,\mathbf{k}$.

Remark:

A vector field is continuous if each of its component function (that is P, Q, R) are continuous.

**EXAMPLE 4** Newton's Law of Gravitation tells you that the magnitude of the force of attraction between two objects of mass m and M is

$$F = \frac{mMG}{r^2}$$

where G is the gravitational constant, and r is the distance between the two objects. Find the vector field describing the gravitational field.

More Examples:

- Force field around an electric charge $Q$ :

$$\vec{F}(\vec{x}) = \frac{\varepsilon_0 q Q}{\|\vec{x}\|^3}\vec{x}$$

- Electric Field around the charge $Q$ :

$$\vec{E}(\vec{x}) = \frac{\vec{F}(\vec{x})}{q} = \frac{\varepsilon_0 Q}{\|\vec{x}\|^3}\vec{x}$$

<u>Gradient Fields.</u>

2D

$$\boxed{\vec{\nabla} f(x,y) = f_x(x,y)\vec{i} + f_y(x,y)\vec{j}}$$

3D

$$\boxed{\vec{\nabla} f(x,y,z) = f_x(x,y,z)\vec{i} + f_y(x,y,z)\vec{j} + f_z(x,y,z)\vec{k}}$$

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**EXAMPLE 6** Find the gradient vector field of $f(x, y) = x^2y - y^3$. Plot the gradient vector field together with a contour map of $f$. How are they related?

## Conservative Vector Fields.

- A vector field $\vec{F}$ is conservative if there is a scalar-valued function $f$ such that

$$\vec{F} = \vec{\nabla} f$$

- The function $f$ is called the potential function of $\vec{F}$.

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EXAMPLE. Show that the Gravitational field is conservative.