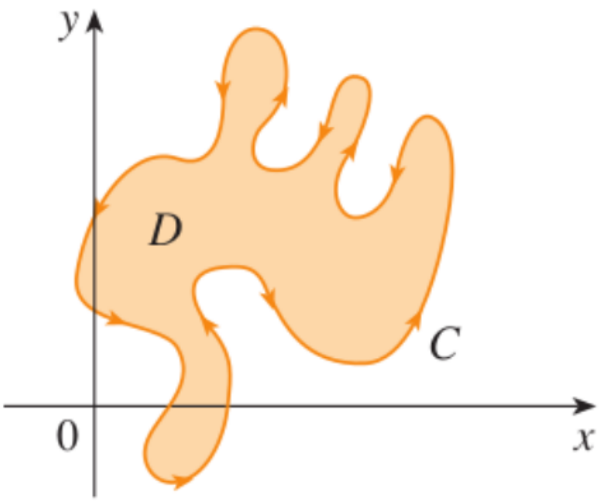
Chapter 16

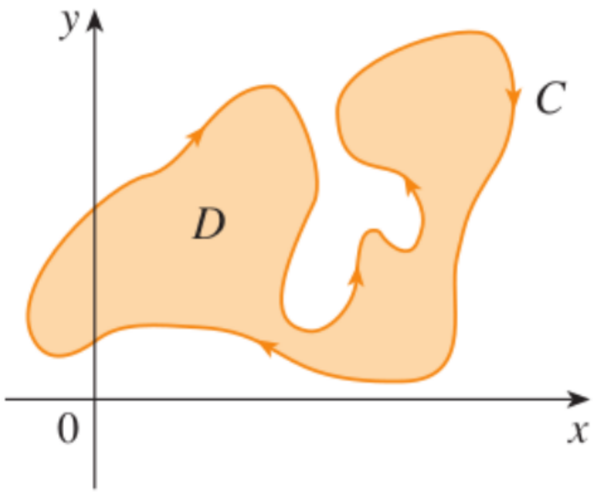
Vector Calculus

16.4 Green's Theorem

Orientation of closed curves



(a) Positive orientation



(b) Negative orientation

EXAMPLE.

Give a parametrization of the positively oriented circle of radius 1 centered at the origin. Find a parametrization giving the negative orientation?

Green's Theorem.

$C:$ closed path with positive orientation.
$D:$ region bounded by C.
If $\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j}$, with P, Q continuously differentiable, then

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Remarks:

- The symbol $\oint_C$ means the path has a positive orientation.
- The left-hand side measures how $\vec{F}$ follows the direction of C.
- The right-hand side measures the tendency of $\vec{F}$ to rotate in the direction of C in the region enclosed by it.

EXAMPLE 1 Evaluate $\oint_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of the line segments from $(0, 0)$ to $(1, 0)$, from $(1, 0)$ to $(0, 1)$, and from $(0, 1)$ to $(0, 0)$.

EXAMPLE 2 Evaluate $\oint_C (3y - e^{\sin x})\,dx + \left(7x + \sqrt{y^4 + 1}\right) dy$, where C is the circle $x^2 + y^2 = 9$.

EXAMPLE 4 Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Computing Areas with Green's Theorem

Recall: $A(D) = \iint_D 1\, dA$

① $A(D) = \oint_C x\, dy.$

② $A(D) = -\oint_C y\, dx.$

③ $A(D) = \frac{1}{2}\left(\oint_C x\, dy - y\, dx\right).$

EXAMPLE 3 Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$