MATH 244

CHAPTER 16

SECTION 16.8: STOKES' THEOREM

CONTENTS

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Curl in 3D

Definition 1. If $\vec{F} = \langle P, Q, R\rangle$ is a vector field in 3D, then

$$\operatorname{curl} \vec{F} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle \, .$$

Another way to write curl $\vec{F}$ is as followed. Define

$$\vec{\nabla} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \quad \Longrightarrow \quad \operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F}.$$

Example 1. Find the curl of $\vec{F} = \langle xz, xyz, -y^2 \rangle$.

Solution.

Theorem 1. Let $\vec{F} = \langle P, Q, R \rangle$. If

- P, Q, R have continuous partial derivatives.
- $\operatorname{curl} \vec{F} = \vec{0}$.

Then $\vec{F}$ is conservative.

Example 2. Let $\vec{F}(x, y, z) = \langle y^2 z^3, 2xyz^3, 3xy^2 z^2 \rangle$.

a) Show that $\vec{F}$ is conservative.

b) Find a function f such that $\vec{F} = \vec{\nabla} f$.

Solution.

Stokes' Theorem

Recall Green's Theorem:

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D Q_x - P_y \, dA,$$

where C is orientated positively. Writing $\vec{F} = \langle P, Q, 0 \rangle$:

$$Q_x - P_y = \langle 0, 0, Q_x - P_y \rangle \cdot \langle 0, 0, 1 \rangle = \operatorname{curl} \vec{F} \cdot \vec{k}$$

so that

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \vec{F} \cdot \vec{k} \, dA.$$

A particular case of Stokes' Theorem.

Theorem 2. Assume

- S be an oriented surface bounded by a loop C with orientation induced by the surface.
- $\vec{F} = \langle P, Q, R \rangle$ with P, Q, R having continuous partial derivatives.

Then,

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}.$$

Rule of Thumb: What we mean by the orientation induced by the surface is: we apply the right-hand rule with the thumb pointing in the direction of the normal vector.

Example 3. Let $\vec{F}(x, y, z) = \langle x + y^2, y + z^2, z + x^2 \rangle$ and C is the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Compute

$$\int_C \vec{F} \cdot d\vec{r}.$$

Solution.

Example 4. Let $\vec{F}(x, y, z) = \langle ze^y, x\cos y, xz\sin y\rangle$ and S be the hemisphere $x^2 + y^2 + z^2 = 16$, $y \geq 0$ oriented in the direction of the positive y-axis. Compute

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}.$$

Solution.