Math 244

Chapter 16

Section 16.9: Divergence Theorem

Contents

Created by: Pierre-Olivier Parisé
Fall 2023

Divergence in 3D

Definition 1. If $\vec{F} = \langle P, Q, R \rangle$ is a vector field in 3D, then

$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Another way to write curl $\vec{F}$ is as followed. Define

$$\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \quad \Longrightarrow \quad \operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F}.$$

Example 1. Find the divergence of $\vec{F} = \langle xz, xyz, -y^2 \rangle$.

Solution.

Theorem 1. Let $\vec{F} = \langle P, Q, R \rangle$ and assume P, Q, R have continuous second partial derivatives. Then

$$\operatorname{div}(\operatorname{curl} \vec{F}) = 0.$$

Example 2. Show that $\vec{F}(x, y, z) = \langle xz, xyz, -y^2 \rangle$ can't be written as the curl of some other vector field.

Solution.

Theorem 2. Assume

- S be a closed surface with positive orientation (outward orientation).
- $\vec{F} = \langle P, Q, R \rangle$ with P, Q, R having continuous partial derivatives.

Then,

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV,$$

where E is the solid bounded by S.

Example 3. Let $\vec{F}(x, y, z) = \langle xye^z, xy^2z^3, -ye^z \rangle$ and S is the surface of the box bounded by the coordinates planes and the planes $x = 3$, $y = 2$, and $z = 1$. Compute the flux of $\vec{F}$ across S.

Solution.