Problem 4

We have

$$\begin{aligned}\int_0^1 \int_y^{2y} \int_0^{x+y} 6xy\, dzdxdy &= \int_0^1 \int_y^{2y} 6xy(x+y)\, dxdy \\ &= \int_0^1 \int_y^{2y} 6x^2y + 6xy^2\, dxdy \\ &= \int_0^1 (2x^3y + 3x^2y^2)\Big|_{x=y}^{x=2y}\, dy \\ &= \int_0^1 (16y^4 + 12y^4) - (2y^4 + 3y^4)\, dy \\ &= \int_0^1 23y^4\, dy \\ &= \frac{23}{5}.\end{aligned}$$

Problem 12

The solid is described in the following way

$$E = \{(x,y,z) \,:\, 0 \leq x \leq \pi,\, 0 \leq y \leq \pi - x,\, 0 \leq z \leq x\}.$$

So,

$$\begin{aligned}\iiint_E \sin y\, dV = \int_0^\pi \int_0^{\pi-x} \int_0^x \sin y\, dzdydx &= \int_0^\pi x\,(-\cos y)\big|_{y=0}^{y=\pi-x}\, dx \\ &= \int_0^\pi -x(1+\cos(\pi - x))\, dx.\end{aligned}$$

After an integration by parts, we get

$$\iiint_E \sin y\, dV = -2 - \pi^2/2 \approx -6.9348.$$

Problem 20

So we have $x^2 + z^2 \leq y \leq 8 - x^2 - z^2$. We have to intersect the two surfaces to find the domain of integration in the XZ-plane. Equating both equations for the surfaces to y, we get

$$x^2 + z^2 = 8 - x^2 - z^2 \iff x^2 + z^2 = 4.$$

So the domain is a circle of radius 2. Thus, the volume will be given by

$$V = \iiint_E dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^{8-r^2} dyrdrd\theta$$

where we describe the domain in the XZ-plane in polar coordinates. So

$$V = 2\pi \int_0^2 (8 - 2r^2) r\, dr = 2\pi \int_0^4 u\, du = 16\pi.$$

Problem 34

We have

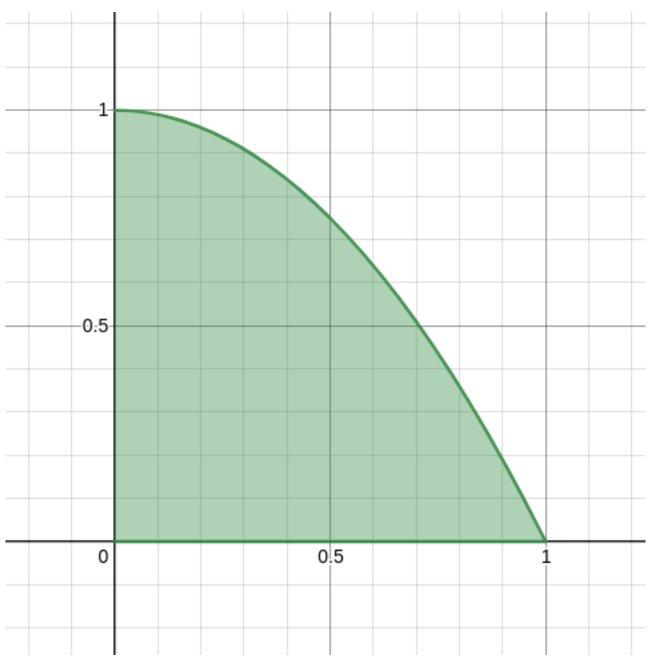
$$E = \{(x, y, z) \,:\, 0 \leq x \leq 1,\, 0 \leq y \leq 1 - x,\, 0 \leq z \leq 1 - x^2\}.$$

The orders we would like are $dzdydx$, $dydxdz$, $dxdydz$, $dzdxdy$, $dxdzdy$.

1. **dzdydx**. Since the bounds depend only on x, we can interchange without problems:

$$\int_0^1 \int_0^{1-x} \int_0^{1-x^2} f(x, y, z)\, dzdydx.$$

2. **dydxdz**. We have to look into the XZ-plane and interchange. The region in this plane are bounded by the curves $x = 0$, $x = 1$, $z = 0$ and $z = 1 - x^2$ and looks like this: So, by seeing



this region as a type two, we get $0 \leq x \leq \sqrt{1 - z}$ and $0 \leq z \leq 1$. We then obtain

$$\int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} f(x, y, z)\, dydxdz.$$

3. **dxdydz**. We have to look into the XY-plane. We see that $0 \leq x \leq \sqrt{1-z}$ and $0 \leq y \leq 1-x$. Here, z is considered as a number which is fixed. If we see this domain as a type II (to interchange the x and the y), we have to deal with two pieces:

   - $0 \leq y \leq 1-\sqrt{1-z}$, then $0 \leq x \leq \sqrt{1-z}$.
   - $1-\sqrt{1-z} \leq y \leq 1$, then $0 \leq x \leq 1-y$.

   So the integral becomes

$$\int_0^1 \left(\int_0^{1-\sqrt{1-z}} \int_0^{\sqrt{1-z}} f(x,y,z)\, dxdy + \int_{1-\sqrt{1-z}}^1 \int_0^{1-y} f(x,y,z)\, dxdy \right) dz.$$

4. **dzdxdy**. We look in the XY-plane in the original configuration. From the bounds in the integrals in x and y, the region in the XY-plane is bounded by the curves $x=0$, $x=1$, $y=0$ and $y=1-x$. So we interchange easily and get $0 \leq x \leq 1-y$ and $0 \leq y \leq 1$ to get

$$\int_0^1 \int_0^{1-y} \int_0^{1-x^2} f(x,y,z)\, dzdxdy.$$

5. **dxdzdy**. We look at the bounds in x and z. We see these bounds give a region bounded by $x=0$, $x=1-y$, $z=0$, and $z=1-x^2$. Again, we have to split into two cases:

   - $0 \leq z \leq 1-(1-y)^2$, $0 \leq x \leq 1-y$;
   - $1-(1-y)^2 \leq z \leq 1$, $0 \leq x \leq \sqrt{1-z}$.

   So the integral in this final order looks like

$$\int_0^1 \left(\int_0^{1-(1-y)^2} \int_0^{1-y} f(x,y,z)\, dxdz + \int_{1-(1-y)^2}^1 \int_0^{\sqrt{1-z}} f(x,y,z)\, dxdz \right) dy.$$