Problem 8

Since $\rho = \sqrt{x^2 + y^2 + z^2}$ and $\cos\phi = z/\rho = z/\sqrt{x^2 + y^2 + z^2}$. Combining everything together gives the following equation:

$$\sqrt{x^2 + y^2 + z^2} = \frac{z}{\sqrt{x^2 + y^2 + z^2}} \quad \Rightarrow \quad x^2 + y^2 + z^2 = z.$$

So, rewritting the last equation in the following way:

$$x^2 + y^2 + (z - 1/2)^2 = 1/4,$$

we see that the surface is a sphere of radius $1/2$ and centered at $(0, 0, 1/2)$.

Problem 26

The equation of the cone is $\phi = \pi/4$ because

$$z = \sqrt{x^2 + y^2} \iff \phi = \pi/4.$$

The equation of the first sphere is $\phi = 1$ and the equation of the second sphere is $\rho = 2$. So the solid can be described in spherical coordinates as followed:

$$E = \{(\rho, \theta, \phi) \,:\, 1 \leq \rho \leq 2, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/4\}.$$

Using the change of variable formula for spherical coordinates, we obtain

$$\begin{aligned}
\iiint_E \sqrt{x^2 + y^2 + z^2}\, dV &= \int_0^{\pi/4} \int_0^{2\pi} \int_1^2 \rho^3 \sin(\phi)\, d\rho d\theta d\phi \\
&= \Big(\int_0^{\pi/4} \sin\phi\, d\phi \Big) \Big(\int_0^{2\pi} d\theta \Big) \Big(\int_1^2 \rho^3\, d\rho \Big) \\
&= (1 - 1/\sqrt{2})(2\pi)(15/4) = (\sqrt{2} - 1)15\pi/(2\sqrt{2}).
\end{aligned}$$

Problem 30

The volume of the solid E is given by

$$V(E) = \iiint_E 1\, dV = \iiint_E dV.$$

The solid lies below the cone $z = \sqrt{x^2 + y^2}$ whose equation in spherical coordinates is $\phi = \pi/4$. Since we want the portion below this cone, the angle ϕ is between $\phi/4$ and $\phi/2$ ($\pi/4 \leq \phi \leq \pi/2$). The equation of the sphere is simply $\rho = 2$. So the description of E in spherical coordinates is

$$E = \{(\rho, \theta, \phi) \,:\, 0 \leq \rho \leq 2, 0 \leq \theta \leq 2\pi, \pi/4 \leq \phi \leq \pi/2\}.$$

Now, using the change of variable formular for spherical coordinates, the volume of E is given by

$$\begin{aligned}
V(E) &= \int_{\pi/4}^{\pi/2} \int_0^{2\pi} \int_0^2 \rho^2 \sin(\phi)\, d\rho d\theta d\phi \\
&= \left(\int_{\pi/4}^{\pi/2} \sin\phi \, d\phi \right) \left(\int_0^{2\pi} \, d\theta \right) \left(\int_0^2 \rho^2 \, d\rho \right) \\
&= (1/\sqrt{2})(2\pi)(8/3) = 16\pi/(3\sqrt{2}).
\end{aligned}$$