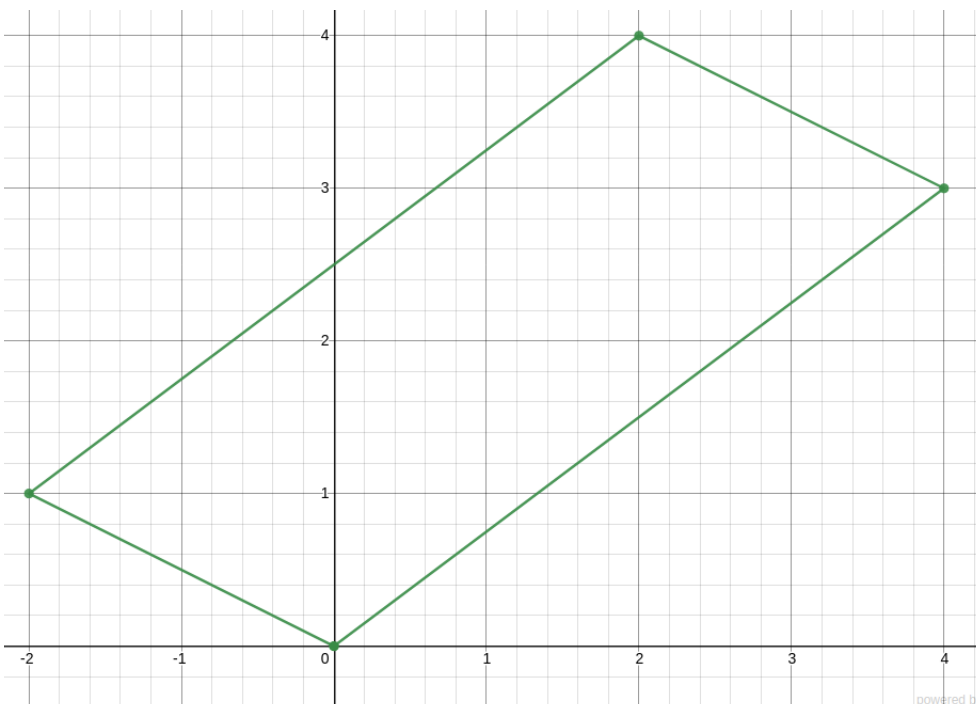
Problem 10

The transformation is given by $T(u,v) = (au, bv)$ and so $x(u,v) = au$ and $y(u,v) = bv$. We then see that $x/a = u$ and $y/b = v$.

The boundary of the region S is the circle $u^2 + v^2 = 1$. Replacing u by x/a and v by y/b, we get the equation $(x/a)^2 + (y/b)^2 = 1$. This is an ellipse centered at the origin. Thus, the region $R = T(S)$ is the interior of the ellipse given by the equation $(x/a)^2 + (y/b)^2 = 1$.

Problem 12

Here's an illustration of the parallelogram.



The equation of the line joining the points $(4,3)$ and $(2,4)$ is $y + x/2 = 5$. The equation of the line joining the points $(-2,1)$ and $(0,0)$ is $y + x/2 = 0$. The equation of the line joining $(2,4)$ and $(-2,1)$ is $y - (3/4)x = 5/2$. The equation of the line joining $(0,0)$ and $(4,3)$ is $y - (3/4)x = 0$.

Take $u = y - (3/4)x$ and $v = y + x/2$. So

- The line passing through $(4,3)$ and $(2,4)$ becomes $0 \leq u \leq 5/2$ and $v = 5$.
- The line passing through $(2,4)$ and $(-2,1)$ becomes $u = 5/2$ and $0 \leq v \leq 5$.
- The line passing through $(-2,1)$ and $(0,0)$ becomes $0 \leq u \leq 5/2$ and $v = 0$.

- The line passing through $(0,0)$ and $(4,3)$ becomes $u=0$ and $0 \le v \le 5$.

These new lines in the uv-plane are the boundary curves of the following rectangle:

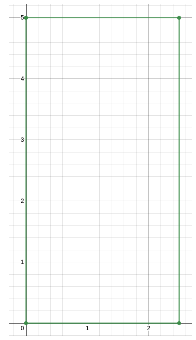


Figure 1: $[0,5/2] \times [0,5]$

Problem 18

The ellipse can be rewritten as

$$x(x-y)+y^2=2.$$

Replacing x and y by the transformations, we have

$$(\sqrt{2}u-\sqrt{2/3}v)(-2\sqrt{2/3}v)+u^2+4uv/\sqrt{3}+2v^2/3=2 \iff u^2+2v^2=2 \iff (u/\sqrt{2})^2+v^2=1.$$

So the region R bounded by the ellipse $x^2+xy+y^2=2$ is the image of the region S bounded by the ellipse $u^2/2+v^2=1$. The description of S is

$$S=\{(u,v)\,:\,u^2/2+v^2\le 1\}.$$

The Jacobian of the transformation is

$$\begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} \sqrt{2} & -\sqrt{2/3} \\ \sqrt{2} & \sqrt{2/3} \end{vmatrix} = 4/\sqrt{3}.$$

So, the integral over R become

$$\iint_R x^2-xy+y^2\,dA=(4/\sqrt{3})\iint_S u^2/2+v^2\,dudv.$$

We will need another change of variable. Take $u=\sqrt{2}r\cos\theta$ and $v=r\sin\theta$. In these coordinates, we see that $0\le r\le 1$ and $0\le\theta\le 2\pi$. The Jacobian of these transformation is

$$\begin{vmatrix} u_r & u_\theta \\ v_r & v_\theta \end{vmatrix} = \begin{vmatrix} \sqrt{2}\cos\theta & -\sqrt{2}r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\sqrt{2}.$$

So, we get

$$\iint_S u^2/2+v^2\,dudv=\int_0^{2\pi}\int_0^1 r^3\sqrt{2}\,drd\theta=\pi\sqrt{2}/2.$$