Problem 4

We have $P(x, y) = y^2 - 2x$ and $Q(x, y) = 2xy$. Then, we see that

$$P_y = 2y \quad \text{and} \quad Q_x = 2y.$$

So $P_y = Q_x$ and the vector field $\vec{F}$ is conservative.

So, there is a function f such that $\vec{F} = \vec{\nabla} f = f_x \vec{i} + f_y \vec{j}$. We must have

$$f_x = y^2 - 2x \quad \text{and} \quad f_y = 2xy.$$

Integrating with respect to x, we find out that $f(x, y) = y^2 x - x^2 + g(y)$. Then, differentiating with respect to y, we find $f_y(x, y) = 2xy + g'(y) = 2xy$ and so $g'(y) = 0$. This implies that $g(y) = C$ for a constant C. Thus,

$$f(x, y) = xy^2 - x^2 + C.$$

Problem 20

If $\vec{r}(t) = (x(t), y(t))$, with $a \leq t \leq b$ is a parametrization of the path C, then the line integral can be expressed as

$$\int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt.$$

where $\vec{F}(x, y) = \langle \sin y, x \cos y - \sin y \rangle$.

Now, it is sufficient to show that $\vec{F}$ is conservative. We have $P(x, y) = \sin y$ and $Q(x, y) = x \cos y - \sin y$, so

$$P_y = \cos y \quad \text{and} \quad Q_x = \cos y.$$

We see that $P_y = Q_x$ and so $\vec{F}$ is conservative.

We now need to find a potential function f. By repeating the procedure from the previous exercise, we obtain

$$f(x, y) = x \sin y + \cos(y).$$

So by the fundamental Theorem for line integrals, we get

$$\int_C \sin y dx + (x \cos y - \sin y)\, dy = f(1, \pi) - f(2, 0) = -1 + 1 = 0.$$