Problem 6

We have $P(x, y) = x^2 + y^2$ and $Q(x, y) = x^2 - y^2$. By Green's Theorem, we have

$$\oint (x^2 + y^2)dx + (x^2 - y^2)dy = \iint_D Q_x - P_y \, dA.$$

The domain D is the triangle with vertices $(0, 0)$, $(2, 1)$, and $(0, 1)$. The positive orientation is a parametrization that passes to all the points in the following order: $(0, 0)$ to $(2, 1)$ to $(0, 1)$ and then coming back to $(0, 0)$. We can write

$$D = \{(x, y) \, : \, 0 \leq x \leq 2, x/2 \leq y \leq 1\}.$$

Thus,

$$\iint_D 2x - 2y \, dA = \int_0^1 \int_{x/2}^1 2x - 2y \, dy dx = 0.$$

So the line integral is zero.

Problem 12

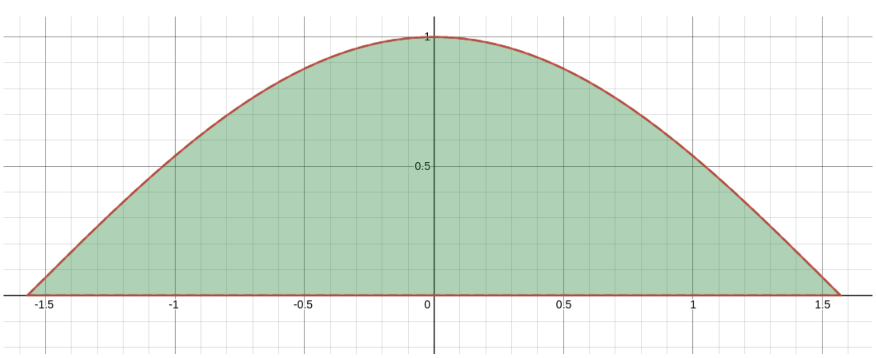
We see that $P(x, y) = e^{-x} + y^2$ and $Q(x, y) = e^{-y} + x^2$ and so

$$Q_x - P_y = 2x - 2y.$$

We want to compute

$$\int_C \vec{F} \cdot d\vec{r} = \int_C e^{-x} + y^2 \, dx + e^{-y} + x^2 \, dy.$$

The curve (in red) and the domain (in green) bounded by the curve is represented below. Taking



the counterclockwise orientation on the curve, we can write

$$\int_C e^{-x} + y^2\,dx + e^{-y} + x^2\,dy = \oint_C e^{-x} + y^2\,dx + e^{-y} + x^2\,dy$$

and by Green's Theorem, we obtain

$$\oint_C e^{-x} + y^2\,dx + e^{-y} + x^2\,dy = \iint_D Q_x - P_y\,dA = \iint_D 2x - 2y\,dA.$$

The description of D is

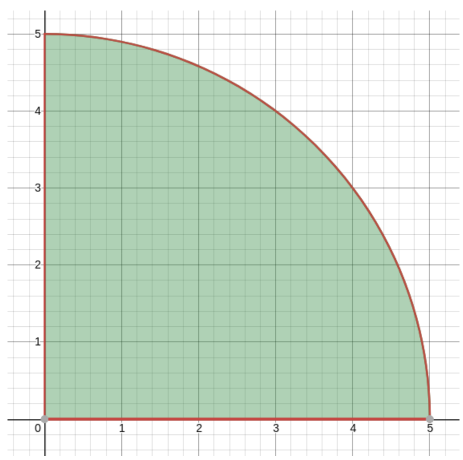
$$D = \{(x, y) \,:\, -\pi/2 \le x \le \pi/2,\, 0 \le y \le \cos x\}.$$

So, we obtain

$$\iint_D 2x - 2y\,dA = \int_{-\pi/2}^{\pi/2} \int_0^{\cos x} 2x - 2y\,dydx = \pi/2.$$

Problem 18

Here is the path traced by the particle and the region D enclosed by the curve C. The work



is given by

$$W = \int_C \vec{F} \cdot d\vec{r}$$

where $\vec{F}(x, y) = \left\langle \sin x, \sin y + xy^2 + \frac{1}{3}x^3 \right\rangle$. Taking the counterclockwise orientation, we see from Green's Theorem that

$$W = \iint_D Q_x - P_y\,dA.$$

We have

$$D = \{(x, y) \,:\, 0 \le x \le 5,\, 0 \le y \le \sqrt{25 - x^2}\},$$

$Q_x = y^2 + x^2$ and $P_y = 0$, so

$$W = \int_0^5 \int_0^{\sqrt{25-x^2}} x^2 + y^2 \, dydx.$$

We change to polar coordinates. We have

$$W = \int_0^5 \int_0^{\pi/2} r^2 \, d\theta dr = \frac{125\pi}{6} \approx 65.4498.$$