

Problem 10

- (a) By definition, $\text{div } \vec{F} = P_x + Q_y + R_z = P_x + Q_y$ because R is independent of z .

If we look only at the x components of $\vec{F}$, that is P , we observe that when y varies, there is no change in P (the first coordinate of the vectors in $\vec{F}$), but when x varies, there is a change in P . This change is positive because when x increases, the values of P increase, this means that $P_x > 0$.

If we look only at the y -components of the vectors in $\vec{F}$, that is Q , we observe that when x varies, there is no change in Q (the second coordinate of the vectors in $\vec{F}$), but when y varies, there is a change in Q . This change is positive because when y increases, the values of Q increase, this means that $Q_y > 0$.

Thus, overall, we have $P_x + Q_y > 0$, meaning that $\text{div } \vec{F} > 0$.

- (b) The fact that $\vec{F}$ doesn't depend on z implies that $\text{curl } \vec{F} = \langle 0, 0, Q_x - P_y \rangle$. Thus, depending on the sign of $Q_x - P_y$, the vector $\text{curl } \vec{F}$ is orthogonal to the XY -plane and points in the direction of the positive z -axis if $Q_x - P_y > 0$ and in the direction of the negative z -axis if $Q_x - P_y < 0$.

Problem 2 (only Q)

We have to check if there are u, v such that $\vec{r}(u, v) = \langle 2, 3, 3 \rangle$.

For the point Q , this means we have to solve the three following equations:

$$1 + u - v = 2, \quad u + v^2 = 3, \quad u^2 - v^2 = 3.$$

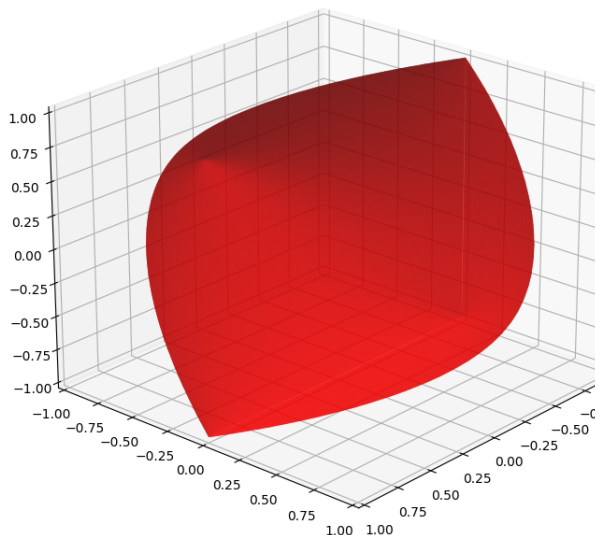
Adding the second equation to the third equation, we obtain

$$u^2 + u = 6 \quad \Rightarrow \quad u^2 + u - 6 = 0 \quad \Rightarrow \quad (u + 3)(u - 2) = 0.$$

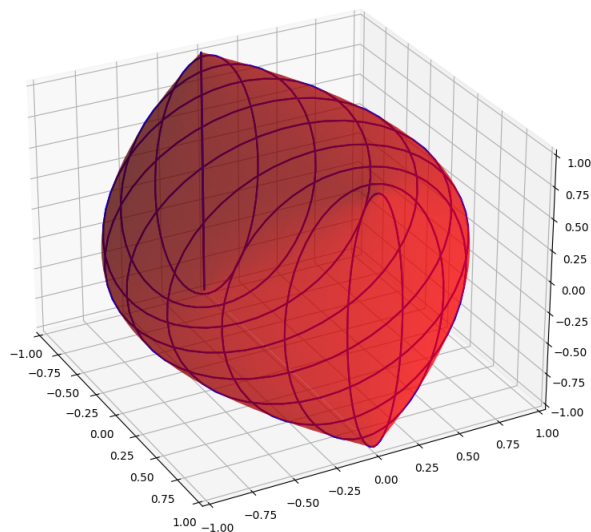
The solutions are $u = -3$ and $u = 2$. we just need one value, say $u = 2$. From the first equation, we see that $u = v$ and so $v = 3$. We just found (u, v) such that $\vec{r}(u, v) = \langle 1, 2, 1 \rangle$ which mean that the point P lies on the surface.

Problem 12

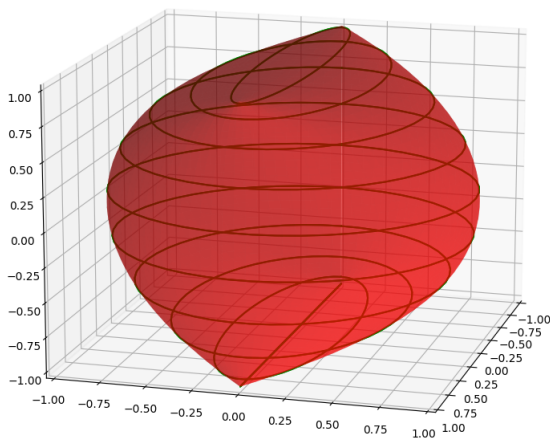
Using either the software on the web, or the python script that I provided, you obtain the following images.



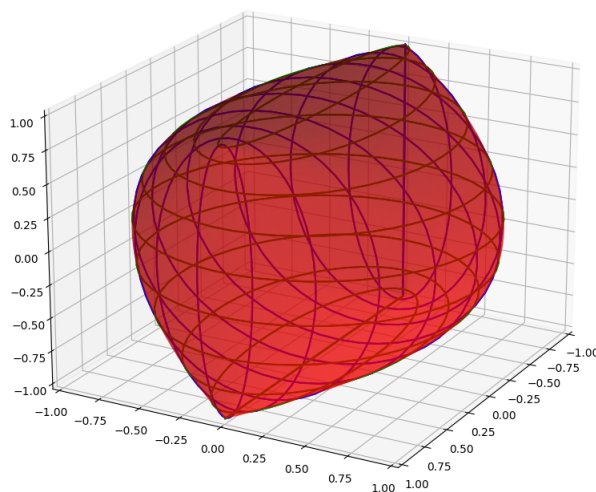
(a) Graph of the surface



(b) Latitudes when u is constant



(c) Longitudes when v is constant



(d) Grid on the surface

This surface is really funny, it looks like a pillow, a really comfortable pillow! :)

Problem 20

The parametric equation is

$$\vec{r}(u, v) = \langle 0, -1, 5 \rangle + u \langle 2, 1, 4 \rangle + \langle -3, 2, 5 \rangle = \langle 2u - 3v, -1 + u + 2v, 5 + 4u + 5v \rangle.$$

Problem 26

The point $(0, 0, 3)$ lies in the plane. The intersection of a plane and a cylinder is a circle in 3D. So the region will be the interior of a circle (but the circle is not parallel to one of the three planes).

An efficient way of solving this problem is to find two orthogonal vector $\vec{a}$ and $\vec{b}$ parallel to the plane such that they belong to the cylinder and then take a linear combinaison $\langle 0, 0, 3 \rangle + u\vec{a} + v\vec{b}$ where $u^2 + v^2 \leq 1$.

A vector parallel to the plane $z = x + 3$ is a vector $\vec{a} = \langle a_1, a_2, a_3 \rangle$ which is orthogonal to the normal vector of the plane. The normal vector of the plane is $\vec{n} = \langle -1, 0, 1 \rangle$. We would also like the tip of the vector $\vec{a}$ belongs to the cylinder, so we also require that $a_1^2 + a_2^2 = 1$. We have to solve

$$\begin{aligned}\vec{a} \cdot \vec{n} &= 0 \\ a_1^2 + a_2^2 &= 1.\end{aligned}$$

This system is explicitly:

$$\begin{aligned}-a_1 + a_3 &= 0 \\ a_1^2 + a_2^2 &= 1.\end{aligned}$$

Since a_2 is free, we may put $a_2 = 0$ and so $a_1 = \pm 1$. We keep $a_1 = 1$ and from the first equation, we get $a_3 = 1$. Our vector is then $\vec{a} = \langle 1, 0, 1 \rangle$.

We have to find a vector $\vec{b} = \langle b_1, b_2, b_3 \rangle$ perpendicular to $\vec{a}$ and lying on the cylinder. These conditions give the following system of equations:

$$\begin{aligned}\vec{b} \cdot \vec{a} &= 0 \\ b_1^2 + b_2^2 &= 1.\end{aligned}$$

Explicitly, it gives the following system of equations:

$$\begin{aligned}b_1 + b_3 &= 0 \\ b_1^2 + b_2^2 &= 1.\end{aligned}$$

A solution to this system is $b_1 = b_3 = 0$ and $b_2 = 1$. So, we obtain $\vec{b} = \langle 0, 1, 0 \rangle$.

Now, we can combine the vectors $\vec{a}$ and $\vec{b}$ with the vector $\langle 0, 0, 3 \rangle$ (the points on the plane), to obtain

$$\vec{r}(u, v) = \langle 0, 0, 3 \rangle + u\vec{a} + v\vec{b}.$$

Since $u^2 + v^2 \leq 1$, we can use polar coordinates $u = \rho \cos \theta$ and $v = \rho \sin \theta$ with $0 \leq \rho \leq 1$ and $0 \leq \theta \leq 2\pi$. Thus, we get, after collecting all the terms together, the following parametrization of the surface:

$$\vec{r}(u, v) = \langle \rho \cos \theta, \rho \sin \theta, 3 + \rho \cos \theta \rangle.$$

There is another solution, which is even simpler. We want the portion of the plane inside the cylinder $x^2 + y^2 = 1$. This means that we want the region $x^2 + y^2 \leq 1$. We can parametrized this region in polar coordinates by setting $x = \rho \cos \theta$ and $y = \rho \sin \theta$ with $0 \leq \rho \leq 1$ and $0 \leq \theta \leq 2\pi$. We can then replace the value of x inside the expression of z to get $z = 3 + \rho \cos \theta$. We then get

$$\vec{r}(\rho, \theta) = \langle \rho \cos \theta, \rho \sin \theta, 3 + \rho \cos \theta \rangle.$$

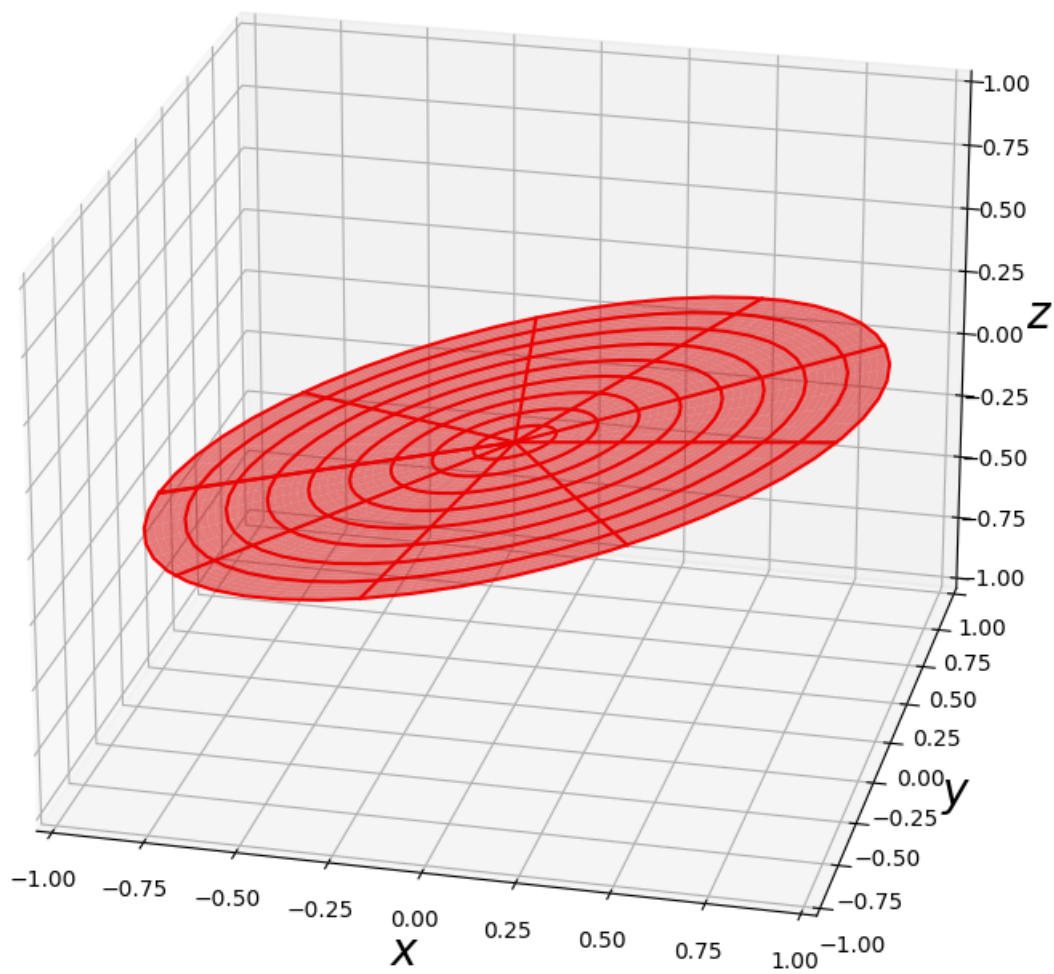


Figure 2: Surface obtained from the parametrization

Problem 38

We have $\vec{r}_u = \langle -2u, 0, -1 \rangle$ and $\vec{r}_v = \langle -2v, -1, 0 \rangle$.

We have to find the point (u_0, v_0) such that $\vec{r}(u_0, v_0) = (-1, -1, -1)$. Analyzing the second and third components, we see that $v = 1$ and $u = 1$. Thus, the tangent vectors at $(-1, -1, -1)$ are

$$\vec{r}_u(1, 1) = \langle -2, 0, -1 \rangle \quad \text{and} \quad \vec{r}_v(1, 1) = \langle -2, -1, 0 \rangle.$$

Thus, the parametric equation of the tangent plane is

$$\vec{r}_{\Pi}(u, v) = \langle -1, -1, -1 \rangle + u\vec{r}_u(1, 1) + v\vec{r}_v(1, 1) = \langle -1 - 2u - 2v, -1 - v, -1 - u \rangle$$

where Π is the name of the plane (the symbol Π is the capital p in greek).

Using python, we obtain the following picture of the tangent plane and the surface.

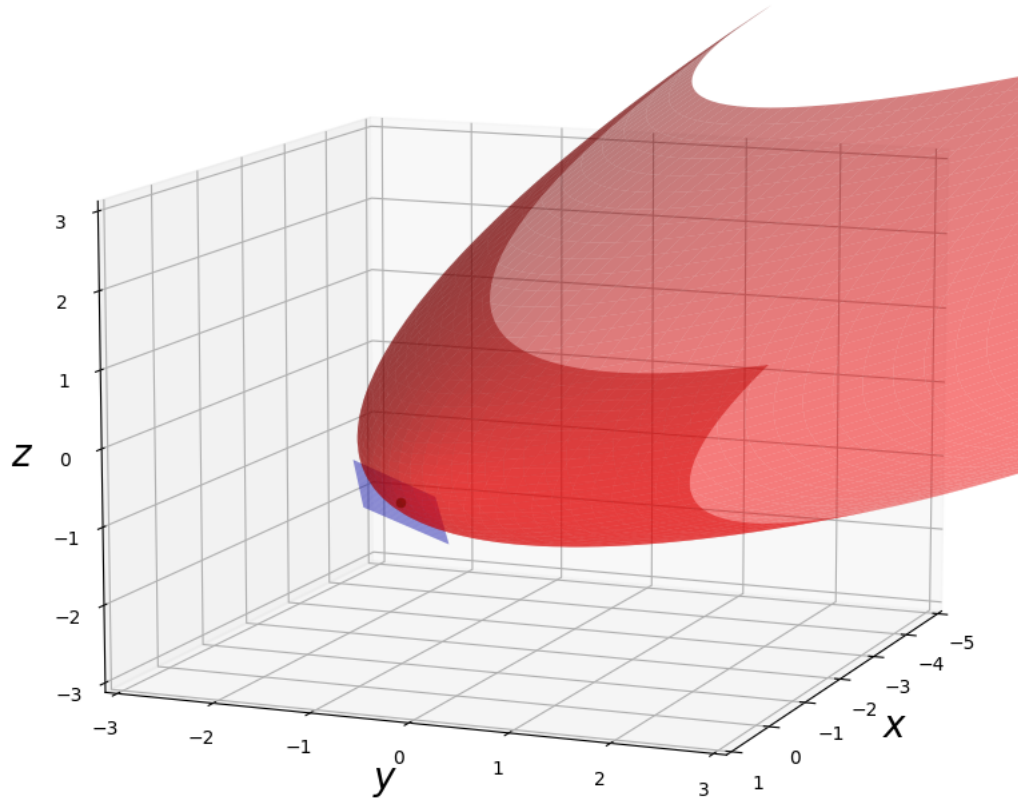


Figure 3: Graph of the surface and its tangent vector