L.1 Mathematical Statements

PROBLEM 1.

a) Yes it is a statement. The statement is false since $|-12| = 12$ (absolute value turns negative numbers into positive numbers).

b) No, this is not a statement. The value of x is not specify, so there is no truth value that can be associated to this sentence.

c) No, this is not a statement. A question is not a statement.

d) Yes, this is a statement. It is true, because assuming that $a = 2$ and $b = 4$, we have $a + b = 2 + 4 = 6$.

L.2 Logic and Mathematical Language

PROBLEM 2.

a) **Converse:** If Angela sleeps in, then it is a Saturday.
Contrapositive: If Angela does not sleep in, then it is not Saturday.

b) **Converse:** If I use my umbrella, then it rains outside.
Contrapositive: If I don't use my umbrella, then it does not rain outside.

PROBLEM 3.

a) The negation is "It is not the case that it is raining and Charlie is cold.". The negation of a statement $P \wedge Q$, is $(\neg P) \vee (\neg Q)$. So, letting P: "It is raining" and Q: "Charlie is cold", a useful reformulation of the negation is "it is not raining or Charlie is not cold".

b) The negation is "It is not the case that if is raining, then Charlie is cold". The negation of a statement $P \Rightarrow Q$ is $P \wedge (\neg Q)$. So, a useful reformulation of the negation is "It is raining and Charlie is not cold".

c) Let's simplify the statement using mathematical symbols. We can equivalently and compactly rewrite the statement as " $\forall x$ real, $\exists y$ real such that $x + y = 0$". The negation is then "It is not the case that $\forall x$ real, $\exists y$ real such that $x + y = 0$". The negation of a universal statement "$\forall x$, $P(x)$" is "$\exists x$, $\neg P(x)$". Let $P(x)$: "$\exists y$ real such that $x + y = 0$". Then we can rewrite the negation of the statement as "$\exists x$ real such that $\neg P(x)$" or

$$\exists x \text{ real such that it is not the case that there exists } y \text{ real such that } x + y = 0.$$

The negation of an existential "$\exists y$, $Q(y)$" is "$\forall y$, $\neg Q(y)$. For a fixed x, let $Q(y)$: "$x + y = 0$". Then we can rewrite the negation of "$\exists y$ real such that $x + y = 0$" as "$\forall x$ real, $\neg Q(y)$", or "$\forall x$ real, $x + y \neq 0$". Therefore, the negation of the whole statement is

$$\exists x \text{ real such that } \forall y \text{ real, } x + y \neq 0\ .$$

PROBLEM 4.

a) By constructing the truth table of $P \Rightarrow Q$ and $Q \Rightarrow P$, show when a conditional statement and its converse do not have the same truth values.

b) By constructing the truth table of $P \Rightarrow Q$ and $(\neg Q) \Rightarrow (\neg P)$, show a conditional statement and its contrapositive always have the same truth values.

a) Here the truth table of $P \Rightarrow Q$ and $Q \Rightarrow P$ combined.

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

We see the truth value differs in the second and first rows.

b) Here the truth table of $P \Rightarrow Q$ and $\neg Q \Rightarrow \neg P$ combined.

P	$\neg P$	Q	$\neg Q$	$P \Rightarrow Q$	$\neg Q \Rightarrow \neg P$
T	F	T	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	T	F	T	T	T

We see that the truth value in every rows are the same.

L.3 Methods of Proof

PROBLEM 5.

a) Assume that a and b are odd integers. By definition, we have $a = 2k + 1$ and $b = 2l + 1$, where k and l are integers. Therefore

$$a + b = 2k + 1 + 2l + 1 = 2(k + l + 1).$$

The last equation expresses $a + b$ as a multiple of 2, so $a + b$ is even.

b) Assume that a is even and that b is odd. By the definitions, we have $a = 2k$ and $b = 2l + 1$, for some integers k and l. Therefore

$$a + b = 2k + 2l + 1 = 2(k + l) + 1.$$

The last equation shows that $a + b$ is an odd number.

PROBLEM 6. We will prove this by contradiction. Assume that $\sqrt{2}$ is a rational number, meaning there are two integers p and q such that $\sqrt{2} = p/q$. We may simplify the fraction p/q so that p and q have no common divisors.

Multiplying by q and squaring both sides of the equation $\sqrt{2} = p/q$ take us to the following equation

$$2q^2 = p^2.$$

This means p^2 is even, so that p is even.

Write $p = 2k$, for some integer k. Replacing the new expression of p is the last equation and after simplify, we obtain

$$q^2 = 2k^2.$$

Therefore, q^2 is even, so that q is even. But if p and q are even, they share a common divisor, that is 2. But we assumed that p and q have no common divisors and this is a contradiction.

Problem 7. Set $m = 3$ and $n = 2$, so that $(2)(3) + (3)(2) = 6 + 6 = 12$.

Problem 8. A sequence (x_n) is a ordered list $(x_1, x_2, x_3, \ldots)$ of real numbers. We say that a sequence (x_n) converges to a real number a if $\forall \varepsilon > 0$, $\exists N \geq 0$ such that $\forall n \geq N$, $|x_n - a| < \varepsilon$.

a) Let $\varepsilon > 0$ be arbitrary. For a fix integer $n \geq 1$, we have

$$\frac{1}{n^2} \leq \frac{1}{n}$$

We have $\frac{1}{n} < \varepsilon$. This is satisfied as long as $n > \frac{1}{\varepsilon}$. Let n_0 be the biggest positive integer less than or equal to $\frac{1}{\varepsilon}$ and define $N = n_0 + 1$.

Let n be an integer such that $n \geq N$. Then, $N > \frac{1}{\varepsilon}$ and therefore $n > \frac{1}{\varepsilon}$. Since $n^2 \geq n$, we obtain that $n^2 > \frac{1}{\varepsilon}$. Thus, $\frac{1}{n^2} < \varepsilon$.

We therefore just showed that for any ε, there exists a positive integer N such that if $n \geq N$, then $\frac{1}{n^2} < \varepsilon$. Hence, (x_n) converges to 0.

b) Just notice that $|\sin(n)/n| \leq \frac{1}{n}$ and the sequence $(1/n)$ converges to 0.