MATH-444	Created by Pierre-O. Parisé
Midterm 01 (50min)	January, Spring 2024

University of Hawai'i

Last name: Solutions.

First name: —

Question:	1	2	3	4	5	6	Total
Points:	10	10	10	10	6	4	50
Score:	—	—	—	—	—	—	—

Instructions:

- Write your complete name on your copy.
- Answer all 6 questions below.
- Write your answers directly on the questionnaire.
- Show ALL your work to have full credit.
- Draw a square around your final answer.
- Return your copy when you're done or at the end of the 2h period.
- No electronic devices allowed during the exam.
- Scientific calculator allowed only (no graphical calculators).
- **Turn off your cellphone(s) during the exam**.
- Lecture notes and the textbook are not allowed during the exam.

Your Signature: —

May the Force be with you!
Pierre

Question 1 **(10 pts)**

Write the complex number $\frac{(1-i)^2}{3+i}i$ in the form $x+iy$.

$$(1-i)^2 = (1-i)(1-i) = 1-i-i+i^2$$

$$= -2i$$

$$\frac{1}{3+i} = \frac{3-i}{|3+i|^2} = \frac{3-i}{10} = \frac{3}{10} - \frac{1}{10}i$$

Hence,

$$\frac{(1-i)^2}{3+i}i = \left(\frac{3}{10} - \frac{1}{10}i\right)(-2i)(i)$$

$$= 2\left(\frac{3}{10} - \frac{1}{10}i\right)$$

$$= \boxed{\frac{3}{5} - \frac{1}{5}i}$$

Question 2 **(10 pts)**

Find the modulus of the complex number $\frac{1+i}{(1-i)(1+3i)}$.

$$\frac{|1+i|}{|1-i||1+3i|} = \frac{1}{|1+3i|} = \boxed{\frac{1}{\sqrt{10}}}$$

QUESTION 3 **(10 pts)**

Find the roots of the following equations: $(z-i)^3 = 8i$.

Set $w = z - i$. Then

$$w^3 = 8i = 8(\cos \pi/2 + i \sin \pi/2)$$

$$\Leftrightarrow \quad w = 2 \cos\left(\frac{\pi/2 + 2k\pi}{3}\right) + i \sin\left(\frac{\pi/2 + 2k\pi}{3}\right)$$

for $k = 0, 1, 2$.

$$\Rightarrow \begin{cases} w_1 = 2\left(\cos(\pi/6) + i \sin(\pi/6)\right) = \sqrt{3} + i \\ w_2 = 2\left(\cos\left(\frac{5\pi}{6}\right) + i \sin\left(\frac{5\pi}{6}\right)\right) = -\sqrt{3} + i \\ w_3 = 2\left(\cos\left(\frac{9\pi}{6}\right) + i \sin\left(\frac{9\pi}{6}\right)\right) = -2i \end{cases}$$

Hence

$$\boxed{\begin{aligned} z_1 &= w_1 + i = \sqrt{3} + 2i \\ z_2 &= w_2 + i = -\sqrt{3} + 2i \\ z_3 &= w_3 + i = -i \end{aligned}}$$

QUESTION 4 **(10 pts)**

Let $f(z) = \frac{z}{z-1}$, with $z \neq 1$. Find the expression of the real and imaginary parts of f in terms of x and y.

Let $w = f(z) = u + iv$ and $z = x + iy$. Then

$$w = \frac{(x+iy)(x-1-iy)}{(x-1)^2 + y^2}$$

$$= \frac{x(x-1) + y^2 - iy}{(x-1)^2 + y^2}$$

$$= \frac{x^2 - x + y^2}{(x-1)^2 + y^2} + i \frac{-y}{(x-1)^2 + y^2}.$$

Hence

$$\boxed{\begin{aligned} \operatorname{Re} f(z) = u(x,y) &= \frac{x^2 - x + y^2}{(x-1)^2 + y^2} \\ \& \\ \operatorname{Im} f(z) = v(x,y) &= \frac{-y}{(x-1)^2 + y^2} \end{aligned}}$$

Question 5 **(6 pts)**

Assume that z_1, z_2 are two complex numbers.
Show that if $\operatorname{Re}(z_1\overline{z_2}) = |z_1||z_2|$, then $\operatorname{Im}(z_1\overline{z_2}) = 0$. Deduce that there exists a positive number α such that $z_1 = \alpha z_2$.

Assume that $\operatorname{Re}(z_1\overline{z_2}) = |z_1||z_2|$. Then

$$\frac{z_1\overline{z_2} + \overline{z_1}z_2}{2} = |z_1||z_2|$$

$$\Rightarrow \left(\frac{z_1\overline{z_2} + \overline{z_1}z_2}{2}\right)^2 = z_1\overline{z_2}\,\overline{z_1}z_2$$

$$\Rightarrow \frac{(z_1\overline{z_2})^2 + 2z_1\overline{z_2}\,\overline{z_1}z_2 + (\overline{z_1}z_2)^2}{4} = \frac{4z_1\overline{z_2}\,\overline{z_1}z_2}{4}$$

$$\Rightarrow \frac{(z_1\overline{z_2})^2 - 2z_1\overline{z_2}\,\overline{z_1}z_2 + (\overline{z_1}z_2)^2}{4} = 0$$

$$\Rightarrow \left(\frac{z_1\overline{z_2} - \overline{z_1}z_2}{2}\right)^2 = 0 \Rightarrow \left(\frac{2i\operatorname{Im} z_1\overline{z_2}}{2}\right)^2 = 0$$

$$\Rightarrow (\operatorname{Im} z_1\overline{z_2})^2 = 0 \Rightarrow \operatorname{Im}(z_1\overline{z_2}) = 0.$$

Assume $z_1 \neq 0$, $z_2 \neq 0$ and let $\beta := z_1\overline{z_2} \in \mathbb{R}^+$.

$$\Rightarrow z_1|z_2|^2 = \beta z_2 \Rightarrow z_1 = \frac{\beta}{|z_2|^2} z_2 = \alpha z_2$$

with $\alpha := \dfrac{\beta}{|z_2|^2} \in \mathbb{R}^+$. $\square$

Question 6 (4 pts)

Answer the following questions with **True** or **False**. Write down you answers on the line at the end of each question. Justify briefly your answer in the space after the statement of the problem.

(a) For any complex numbers z_1 and z_2, we have $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$. (/ 1)

$$\left.\begin{array}{l} z_1 = \cos(\pi) \\ z_2 = \cos(\pi) \end{array}\right\} \Rightarrow z_1 z_2 = \cos(2\pi) = \cos(0).$$

$$\Rightarrow \operatorname{Arg}(z_1 z_2) = 0 \neq 2\pi = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2).$$

(a) False.

(b) If $f(z) = z^4$, then $\rho(\theta, r) = r^4$ and $\phi(\theta, r) = 4\theta$. (/ 1)

$$z = r(\cos\theta + i\sin\theta)$$

$$\Rightarrow \rho = r^4 \quad \text{and} \quad \phi = 4\theta.$$

(b) True.

(c) If $|z - 3i| \leq 1$, then $|z - 4| \leq 6$. (/ 1)

$$|z-4| \leq |z-3i| + |3i-4|$$

$$\leq 1 + 5 = 6$$

(c) True.

(d) If $z = i$, then $i^{4n} = 1$, for any integer n. (/ 1)

$$i^{4n} = (i^4)^n = (i^2 i^2)^n = 1^n = 1, \quad n \geq 0$$

$$i^{-4k} = \frac{1}{i^{4n}} = \frac{1}{1} = 1, \quad k > 0.$$

(d) True.