MIDTERM 02 M444 S2024

Solutions Pierre Parise

Question 1

(a) Notice that $|a_n| \leq \frac{1}{n}$. Hence

$\boxed{a_n \to 0}$ because $\frac{1}{n} \to 0$.

(b) $\left|\frac{1+i}{2}\right| = \frac{\sqrt{2}}{2} < 1$. Therefore

$$\sum_{n=0}^{\infty} \left(\frac{1+i}{2}\right)^n = \frac{1}{1 - \frac{1+i}{2}} = \boxed{\frac{2}{1-i}}$$

Question 2

Write $i = e^{i\pi/2}$. So

$$e^x e^{iy} = e^{i\pi/2} \implies x = 0, \ y = \frac{\pi}{2} + 2k\pi$$

Hence, $\boxed{z = i\left(\frac{\pi}{2} + 2k\pi\right), \ k \in \mathbb{Z}.}$

Question 3

(a) $$f(z) = (x+iy)(\sin x \cosh y + i \cos x \sinh y)$$

$$= \underbrace{(x \sin x \cosh y - y \cos x \cosh y)}_{=u}$$

$$+ i \underbrace{(x \cos x \sinh y + y \sin x \cosh y)}_{=v}$$

(b) $$f(z) = \cos(x^2 - y^2 + 2xyi)$$

$$= \underbrace{\cos(x^2-y^2)\cosh(2xy)}_{=u} - i \underbrace{\sin(x^2-y^2)\sinh(2xy)}_{=v}$$

Question 4

(a) $$\mathrm{Log}(1+i) = \log|1+i| + i\,\mathrm{Arg}(1+i)$$

$$= \log\sqrt{2} + i\frac{\pi}{4}.$$

(b) $$i^i = e^{i \log i} = e^{i\{i\frac{\pi}{2} + 2k\pi i \,:\, k \in \mathbb{Z}\}}$$

$$= e^{\{-\pi/2 - 2k\pi \,:\, k \in \mathbb{Z}\}}$$

$$\Rightarrow \boxed{i^i = \{e^{-\pi/2 + 2k\pi} : k \in \mathbb{Z}\}.}$$

Question 5

(a) $\cosh(-z) = \dfrac{e^{-z} + e^{-(-z)}}{2} = \dfrac{e^{-z} + e^{z}}{2} = \cosh(z).$

(b) $\sinh(-z) = \dfrac{e^{-z} - e^{-(-z)}}{2} = \dfrac{e^{-z} - e^{z}}{2} = -\sinh(z).$

(c) Let $z = x + iy \rightarrow \bar{z} = x - iy$. Then

$$\begin{aligned}
\cos(\bar{z}) &= \cos(x)\cosh(-y) - i\sin(x)\sinh(-y) \\
&\overset{(a)\,\&\,(b)}{=} \cos(x)\cosh(y) + i\sin(x)\sinh(y) \\
&= \overline{\cos(x)\cosh(y) - i\sin(x)\sinh(y)} \\
&= \overline{\cos(z)}.
\end{aligned}$$

$\square$

Question 6

(a) False, $z = w = -1$.

(b) False, $z = -1$.

(c) False, $z = 2\pi i$.

(d) False, $(1+i)^i = e^{i\log(1+i)}$ and $\log(1+i)$ is multi-valued.