MATH-444
Midterm 03 (50min)

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Last name: Solution

First name: —

Question:	1	2	3	4	5	Total
Points:	10	10	20	6	4	50
Score:	–	–	–	–	–	–

Instructions:

- Write your complete name on your copy.
- There are 5 questions on the exam.
- Write your answers directly on the questionnaire.
- Show ALL your work to have full credit.
- Draw a square around your final answer.
- Return your copy when you're done or at the end of the 50min period.
- No electronic devices allowed during the exam.
- Scientific calculator allowed only (no graphical calculators).
- **Turn off your cellphone(s) during the exam**.
- Lecture notes and the textbook are not allowed during the exam.

Your Signature: —

May the Force be with you!
Pierre

Question 1 (10 pts)

Find the value of the following limits, if the limit exists. If the limit does not exist, explain why.

(a) (5 points) $\lim_{z\to 0} z\operatorname{Arg}(z)$.

(b) (5 points) $\lim_{z\to 0} \frac{\overline{z}}{z}$.

(a) We have $|\operatorname{Arg}(z)| \le \pi$. Since $z \to 0$ as $z \to 0$, by Squeeze Theorem

$$\lim_{z\to 0} z \operatorname{Arg} z = 0.$$

(b) $z = x \to 0$, then

$$\lim_{z\to 0} \frac{\overline{z}}{z} = \lim_{x\to 0} \frac{x}{x} = 1$$

Let $z = iy \to 0$. then

$$\lim_{z\to 0} \frac{\overline{z}}{z} = \lim_{y\to 0} \frac{-iy}{iy} = -1.$$

So, $\lim_{z\to 0} \frac{\overline{z}}{z} \nexists$.

Question 2 **(10 pts)**

Using the Cauchy-Rieman Equations, determine if the following function is analytic on its domain:

$$f(z) = z^3 = (x^3 - 3xy^2) + i(3x^2y - y^3).$$

The function f is defined $\forall z \in \mathbb{C}$.

We have $u(x,y) = x^3 - 3xy^2$

$v(x,y) = 3x^2y - y^3$.

① $\left.\begin{array}{l} u_x = 3x^2 - 3y^2 \\ v_y = 3x^2 - 3y^2 \end{array}\right\} u_x = v_y$

② $\left.\begin{array}{l} u_y = -6xy^2 \\ v_x = 6xy \end{array}\right\} u_y = -v_x$.

The Cauchy-Riemann Equations are satisfied throughout $U = \mathbb{C}$

$\Rightarrow f$ is analytic on $\mathbb{C}$.

Question 3 (20 pts)

Compute the value of the path integrals.

(a) (10 points) $\displaystyle\int_{[z_2,z_1]} \overline{z}\,dz$, where $[z_2, z_1]$ is the line segment joining $z_1 = -2$ and $z_2 = 1$.

(b) (10 points) $\displaystyle\int_{[z_1,z_2,z_3]} ze^z\,dz$, where $z_1 = \pi$, $z_2 = -1$, and $z_3 = -1 - i\pi$.

(a) $[z_2, z_1] = \{ (1-t) z_2 + t z_1 \;:\; 0 \le t \le 1 \}$.

So, $dz = (-z_2 + z_1)$

$$\Rightarrow \int_{[z_2,z_1]} \overline{z}\,dz = \int_0^1 \overline{(1-t)z_2 + t z_1}\,(-z_2+z_1)\,dt$$

$$= (-1-2)\int_0^1 (1-t)(1) + t(-2)\,dt$$

$$= (-3)\int_0^1 1-3t\,dt = \boxed{\frac{3}{2}}$$

(b) $F(z) = ze^z - e^z \Rightarrow F'(z) = ze^z$

$$\Rightarrow \int_{[z_1,z_2,z_3]} ze^z\,dz = F(z_3) - F(z_1)$$

$$= (-1-i\pi)e^{-1-i\pi} - e^{-1-i\pi}$$

$$- \pi e^{\pi} + e^{\pi}$$

$$= (1+i\pi)e^{-1} + e^{-1} - \pi e^{\pi} + e^{\pi}$$

$$= \boxed{\frac{(2+i\pi)}{e} + (1-\pi)e^{\pi}}$$

QUESTION 4 **(6 pts)**

Let $C_R(z_0)$ be the positively oriented circle with center z_0 and radius $R > 0$.

(a) (3 points) Compute the path integral $\int_{C_R(z_0)} \operatorname{Im} z \, dz$.

(b) (3 points) Use part (a) to show that $f(z) = \operatorname{Im} z$ has no antiderivative on any open subset of $\mathbb{C}$.

(a) Write $\operatorname{Im} z = \frac{z - \bar{z}}{2i}$. Then

$$\int_{C_R(z_0)} \frac{\operatorname{Im} z}{2i} \, dz = \frac{1}{2i}\left(\int_{C_R(z_0)} z \, dz - \int_{C_R(z_0)} \bar{z} \, dz \right)$$

$$= -\frac{1}{2i} \int_0^{2\pi} (\bar{z}_0 + re^{-i\theta}) ire^{i\theta} \, d\theta$$

$$= -\frac{1}{2\pi} \int_0^{2\pi} r^2 \, d\theta = -r^2 \pi. \quad (*)$$

(b) Assume that there is an F, analytic on U s.t. $F'(z) = \operatorname{Im} z$.

Let $\overline{B_R(z_0)} \subset U$, for some $z_0 \in U$, $R > 0$.

then,
$$\int_{C_R(z_0)} \operatorname{Im} z \, dz = \int_{C_R(z_0)} F'(z) \, dz = 0$$

because $C_R(z_0)$ is closed. This contradicts $(*)$.

So, $\operatorname{Im} z$ can't have an antiderivative on any open set $U \subset \mathbb{C}$. $\square$

QUESTION 5 **(4 pts)**

Answer the following questions with **True** or **False**. Write down you answers on the line at the end of each question. Justify briefly your answer in the space after the statement of the problem.

(a) The function Log(z) is analytic on $\mathbb{C}\backslash\{0\}$. (/ 1)

Not analytic on $(-\infty, 0]$.

(a) False.

(b) If F is an entire antiderivative of a function f, then $\int_\gamma f(z)\,dz = 0$. (/ 1)

$f(z) = z$ and $\gamma = [0, i]$.

$$\int_\gamma z\,dz = \frac{z^2}{2}\Big|_0^i = -\frac{1}{2} \neq 0$$

(b) False.

(c) For any $R > 0$ and any $n \in \mathbb{Z}$, we have $\int_{C_R(0)} \frac{1}{z^n}\,dz = 0$. (/ 1)

$$\int_{C_1(0)} \frac{1}{z}\,dz = 2\pi i \neq 0.$$

(c) False.

(d) If f is analytic on an open set U and $f'(z) = 0$ for any $z \in U$, then f is the constant function. (/ 1)

U should be simply connected.

Ex.: $f \equiv 1$ on $B_{1/2}(-1)$

$f \equiv -1$ on $B_{1/2}(1)$.

(d) False.