Section 1.1: Complex Numbers

DEF.
- A complex number $z = (x, y)$, $x, y \in \mathbb{R}$.
- The set of z is denoted by $\mathbb{C}$.
- x: called the real part.
- y: called the imaginary part.

DEF. Let $z = (x, y)$ and $w = (s, t)$.

1) $z = w \iff x = s$ and $y = t$.
2) Sum:
$$z + w := (x + s,\ y + t).$$
3) Difference:
$$z - w := (x - s,\ y - t).$$
4) Product:
$$zw := (xs - yt,\ xt + ys).$$
5) Complex conjugate:
$$\bar{z} = (x, -y).$$

THM $\forall z_1, z_2, z_3 \in \mathbb{C}$

a) $z_1 + z_2 = z_2 + z_1$ (Commutativity of $+$).
b) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ (Asso. of $+$).
c) $\exists!\ 0 = (0, 0)$ s.t. $0 + z = z + 0 = z$, $\forall z \in \mathbb{C}$.
d) The additive inverse of $z = (x, y)$ is $-z = (-x, -y)$. ($z + (-z) = 0$).
e) $z_1 z_2 = z_2 z_1$ (Comm. of Product).
f) $(z_1 z_2) z_3 = z_1 (z_2 z_3)$ (Assoc. of Product).
g) $z_1 (z_2 + z_3) = z_1 z_2 + z_1 z_3$ (Distr. $\bullet$ over $+$).

h) The multiplicative identity is $1 := (1, 0)$
$(1z = z, \forall z \in \mathbb{C})$.

i) For every $z \neq 0$,

$$z^{-1} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2} \right) \left(\frac{\bar{z}}{z\bar{z}} \right).$$

This means:

$$z z^{-1} = 1.$$

We also write $z^{-1} = 1/z$.

PROOF e) $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$.
So,

$$z_1 z_2 = (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2)$$

$$z_2 z_1 = (x_2 x_1 - y_2 y_1, y_2 x_1 + x_2 y_1)$$

From commutativity of multiplication of $\mathbb{R}$ numbers,

$$z_1 z_2 = z_2 z_1.$$

i) Let $z = (x, y)$. then

$$z \cdot z^{-1} = (x, y) \cdot \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2} \right)$$

$$= \left(\frac{x^2}{x^2+y^2} - \frac{y(-y)}{x^2+y^2}, \frac{x(-y)}{x^2+y^2} + \frac{yx}{x^2+y^2} \right)$$

$$= \left(\frac{x^2+y^2}{x^2+y^2}, \frac{-xy+xy}{x^2+y^2} \right) = (1, 0) = 1 \quad \square$$

REMARK 1) For any $x \in \mathbb{R}$, $x \sim (x, 0)$.

2) From the def. of the product:

$$(0,1)\cdot(0,1) = (-1, 0) \sim -1$$

We define $i := (0,1)$

3) Using the algebraic properties:

$$z = (x,y) = (x,0) + (0,y)$$

$$= (x,0) + (y,0)(0,1)$$

$$= x + yi = x + iy.$$

This is the cartesian form of z.

Ex: $(1+i) + (2-i) = 3 + (0)i = 3$

$(1+i)(2-i) = 2 - i + 2i - \underbrace{i^2}_{=-1} = 3 + i.$

4) $\bar{z} = x - iy$

5) $\forall z \in \mathbb{C}$, $z \neq 0$, $z^{-1} = \dfrac{x}{x^2+y^2} - i\dfrac{y}{x^2+y^2}$

6) If $z, w \in \mathbb{C}$ with $w \neq 0$,

$$\frac{z}{w} = z \cdot \frac{1}{w} = z \cdot w^{-1} = \frac{z \cdot \bar{w}}{w \cdot \bar{w}}$$

7) Purely real number: $z = x$ (no imaginary part)

8) Purely imaginary number: $z = iy$

(no real part).

DEF. Let $z = x+iy$ and $n>0$ be an integer.

1) z^n is defined as
   - $n=1: \quad z^1 = z$
   - $n>1: \quad z^n = z^{n-1} z = \underbrace{z \cdot z \cdots z}_{n \text{ times}}$.

2) $z^{-n} = \dfrac{1}{z^n} \left(= \dfrac{\bar{z}^n}{\overline{z^n \bar{z}^n}} \right)$

3) $z \neq 0, \quad z^0 = 1$.

Prop.: $z^m z^n = z^{m+n}$.

$z^{mn} = (z^m)^n$.

THM. Let $z; z_1; z_2 \in \mathbb{C}$. Then

a) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

b) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$

c) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$

d) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}, \ z_2 \neq 0$

e) $\overline{(z^n)} = \bar{z}^n, \ n \geq 0$

f) $\bar{\bar{z}} = z$

g) $z + \bar{z} = 2 \operatorname{Re} z$

h) $z - \bar{z} = 2i \operatorname{Im} z$

PROOF We prove c), d) and g)

c) $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$$= x_1 x_2 - y_1 y_2 + (x_1 y_2 + x_2 y_1) i$$

and

$\bar{z}_1 \bar{z}_2 = (x_1 - iy_1)(x_2 - iy_2)$

$$= x_1 x_2 - y_1 y_2 + (x_1(-y_2) + x_2(-y_1)) i$$

$$= x_1x_2 - y_1y_2 - (x_1y_2 + x_2y_1)\,i$$

$$= \overline{z_1}\,\overline{z_2}\ .$$

d) Notice that if $z \neq 0$, then $z \cdot z^{-1} = 1$

$$\Rightarrow \quad \overline{z \cdot z^{-1}} = \overline{1} = 1$$

$$(c) \quad \Rightarrow \quad \overline{z} \cdot \overline{z^{-1}} = 1 \quad \Rightarrow \quad \overline{z^{-1}} = \overline{\left(\frac{1}{z}\right)} = \frac{1}{\overline{z}}$$

Now,

$$\overline{\left(\frac{z_1}{z_2}\right)} = \overline{z_1 \cdot z_2^{-1}} \overset{(c)}{=} \overline{z_1}\ \overline{z_2^{-1}} = \frac{\overline{z_1}}{\overline{z_2}}\ .$$

g) Let $z = x + iy$.

$$z + \overline{z} = (x+iy) + (x - iy)$$

$$= 2x + 0i = 2x = 2\operatorname{Re} z\ . \quad \square$$