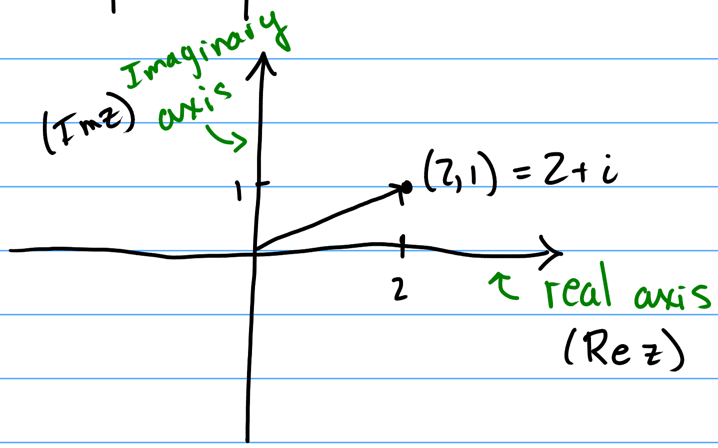
Section 1.2: The Complex Plane

Complex plane $\longleftrightarrow$ Cartesian plane



ABSOLUTE VALUE

DEF If $z = x+iy$, then the absolute value or the norm or the modulus of z is

$$|z| = \sqrt{x^2 + y^2}.$$

Note: • if $z = x$, then $|z| = \sqrt{x^2} = |x|$ ← regular abs. value of $\mathbb{R}$.

• $|z| = 0 \iff z = 0$.

• $|-z| = |z|$

DEF. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then the distance between z_1 and z_2 is

$$d(z_1, z_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= |z_1 - z_2| \;\; (= |z_2 - z_1|)$$

Ex. 1.2.4 (a) Find the set of complex numbers z s.t.

$$|z+4-i| = 2.$$

(b) Find the set of complex numbers z s.t.

$$|z+4-i| \leq 2.$$

Sol. (a) Let $z = x+iy$. So

$$|z+4-i| = 2 \iff \sqrt{(x+4)^2+(y-1)^2} = 2$$

$$\iff (x+4)^2+(y-1)^2 = 4$$

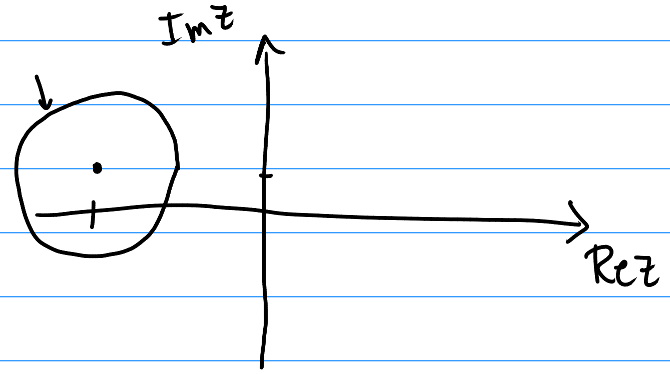
$$\left((x-a)^2 + (y-b)^2 = R^2\right)$$

A circle of radius 2 centered at $(-4,1)$.

The set of z s.t.

$$|z-z_0| = R$$

is a circle centered at z_0 and of radius R

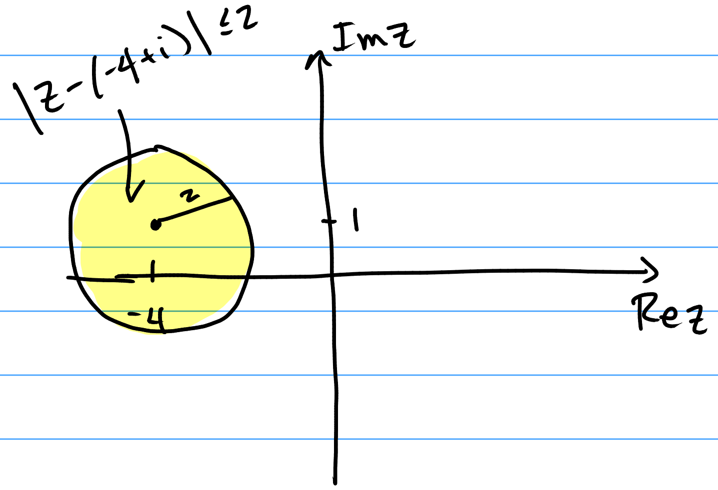


(b) $|z+4-i| \leq 2 \iff |z+4-i|^2 \leq 4$

$$\iff (x+4)^2 + (y-1)^2 \leq 4.$$

A disk of radius 2 and centered at $(-4,1)$.

In general, $|z - z_0| \le R$ is a disk of radius R and centered at z_0.



Prop. 1.2.5 Let $z, z_1, z_2, \ldots, z_n$ be complex numbers.

(a) $|z| = \sqrt{z\bar{z}}$ or $|z|^2 = z\bar{z}$

(b) $|\bar{z}| = |z|$

(c) $|z_1 z_2| = |z_1||z_2|$ (d) $|z_1 z_2 \cdots z_n| = |z_1||z_2|\cdots|z_n|$

(e) $|z^n| = |z|^n$, for any $n \geq 0$

(f) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ if $z_2 \neq 0$.

PROOF. Prove (c). Here,

$$
\begin{aligned}
|z_1 z_2|^2 &\overset{(a)}{=} (z_1 z_2)\overline{(z_1 z_2)} = z_1 z_2 \bar{z}_1 \bar{z}_2 \\
&= z_1\bar{z}_1 \, z_2\bar{z}_2 \\
&\overset{(a)}{=} |z_1|^2 |z_2|^2 \quad (\text{take } \sqrt{\cdot}\,)
\end{aligned}
$$

$\square$

Example 1.2.7 Compute

$$\left| \frac{(3+4i)^2 (3-i)^{10}}{(3+i)^9} \right| = A$$

Sol.

$$A = \frac{\left|(3+4i)^2 (3-i)^{10}\right|}{\left|(3+i)^9\right|}$$

$$= \frac{\left|(3+4i)^2\right| \left|(3-i)^{10}\right|}{|3+i|^9}$$

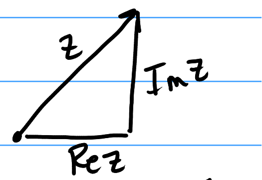
$$= \frac{|3+4i|^2 \, |3-i|^{10}}{|3+i|^9} = \frac{|3+4i|^2 \, |3+i|^{10}}{|3+i|^9}$$

$$= |3+4i|^2 \, |3+i|$$

$$= \boxed{25\sqrt{10}}$$

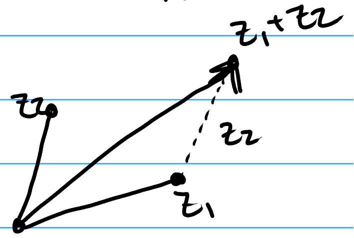
Prop. 1.2.8 Let $z, z_1, z_2, \ldots, z_n$ be complex numbers. Then

(a) $|\mathrm{Re}\, z| \le |z|$, $|\mathrm{Im}\, z| \le |z|$



$$|z| \le |\mathrm{Re}\, z| + |\mathrm{Im}\, z|$$

(b) $|z_1 + z_2| \le |z_1| + |z_2|$



(TRIANGLE INEQUALITY)

or

$$|z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|$$

(c) $|z_1 - z_2| \le |z_1| + |z_2|$

(d) $|z_1 \pm z_2| \geq ||z_1| - |z_2||$

PROOF. (d) We are going to prove that

$$|z_1 + z_2| \geq \big|\, |z_1| - |z_2| \,\big| = \pm\left(|z_1| - |z_2| \right)$$

We have

$$|z_1| = |z_1 + z_2 - z_2|$$

$$\leq |z_1 + z_2| + |-z_2| = |z_1 + z_2| + |z_2|$$

$$\Rightarrow \quad |z_1| - |z_2| \leq |z_1 + z_2| \quad ①$$

On the other hand:

$$|z_2| \leq |z_2 + z_1| + |z_1|$$

$$\Rightarrow \quad |z_2| - |z_1| \leq |z_1 + z_2| \quad ②$$

So, combining ① + ②

$$\Rightarrow \quad |z_1 + z_2| \geq \pm\left(|z_1| - |z_2|\right) = \big|\, |z_1| - |z_2| \,\big|. \qquad \square$$