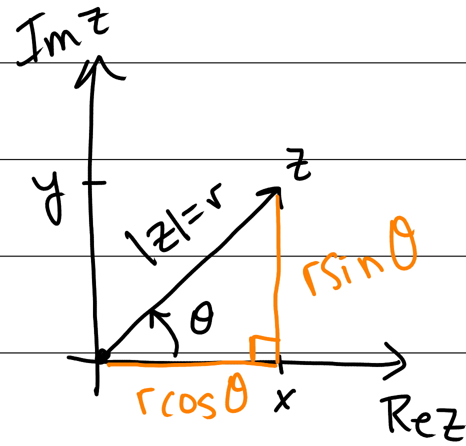
SECTION 1.3: POLAR FORM



DEF. Let $z = x + iy$, $z \neq 0$.
Let
$$r := \sqrt{x^2 + y^2} = |z|$$
and θ be an angle such that
$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$
$$\left(\frac{x}{r} = \cos\theta\right) \qquad \left(\frac{y}{r} = \sin\theta\right).$$

The polar form (or representation) of z is
$$z = r(\cos\theta + i\sin\theta)$$
$$= r\cos\theta + i r\sin\theta.$$
Here r: modulus.
θ: argument.

Remark • The polar representation of 0 is undefined because $\arg z$ is undefined.

• We can find the argument θ by using the $\tan^{-1}$:
$$\theta = \tan^{-1}(y/x), \quad x \neq 0.$$

• $|z| = 1 \Rightarrow z = \cos\theta + i\sin\theta$.
These numbers are called unimodular.

DEF 1.3.2 The principal value of the argument of $z = x+iy$ $(z \neq 0)$ is the unique number $\mathrm{Arg}(z)$ such that

- $-\pi < \mathrm{Arg}(z) \leq \pi$
- $\cos(\mathrm{Arg}\, z) = x/r$
- $\sin(\mathrm{Arg}\, z) = y/r$

} $\mathrm{Arg}(z)$ is an argument for z.

The set of all arguments is

$$\arg(z) = \{ \mathrm{Arg}(z) + 2k\pi : k \in \mathbb{Z} \}.$$

Remark Here $\arg(z)$ is multi-valued.

Example 1.3.3 Find the modulus, the argument and polar form of

(d) $z_4 = 1+i$ (e) $z_5 = 1-i$ (f) $z_6 = -1-i$

Sol. (d) $|z_4| = \sqrt{2}$.

$$\arg(z) = \{ \mathrm{Arg}(z) + 2k\pi : k \in \mathbb{Z} \}.$$

Here,

$$\theta = \tan^{-1}(1/1) = \pi/4 = \mathrm{Arg}(z)$$

$$\Rightarrow z_4 = \sqrt{2}\left(\cos(\pi/4) + i \sin(\pi/4) \right).$$

(e) $|z_5| = \sqrt{2}$

Here $\theta = \tan^{-1}(-1/1) = -\pi/4 = \text{Arg}(z)$

$$\Rightarrow \quad z_5 = \sqrt{2}\left(\cos\left(-\pi/4\right) + i\sin\left(-\pi/4\right)\right)$$

(f) $|z_6| = \sqrt{2}$

Here

$$\theta = \tan^{-1}(-1/-1) + \pi = \frac{\pi}{4} + \pi = \frac{5\pi}{4}$$

Here,

$$\text{Arg}(z) = \frac{5\pi}{4} - 2\pi = -\frac{3\pi}{4}$$

So,

$$z_6 = \sqrt{2}\left(\cos\left(5\pi/4\right) + i\sin\left(5\pi/4\right)\right).$$

Remarks

$$\arg(\bar{z}) = -\arg(z)$$

$$\arg(-z) = \arg(z) + \pi.$$

Multiplication in polar form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$
$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$z_1 z_2 = r_1 r_2 (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i\, r_1 r_2 (\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2)$$

Recall:
$$\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 = \cos(\theta_1 + \theta_2)$$
$$\cos\theta_1 \sin\theta_2 + \sin\theta_1 \cos\theta_2 = \sin(\theta_1 + \theta_2)$$

Polar form of the product

$$\Rightarrow z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2))$$

Conse.: $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.

Polar form of the inverse:
$$\frac{1}{z_1} = \frac{1}{r_1}(\cos(-\theta_1) + i\sin(-\theta_1))$$

$$= \frac{1}{r_1}\left(\cos\theta_1 - i\sin\theta_1\right)$$

Conse: $\arg\left(\frac{1}{z_1}\right) = -\arg(z_1) = \arg(\bar{z}_1)$

Polar form of quotient: $(z_2 \neq 0)$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right).$$

Conse.: $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$.

De Moivre's Identity

Let $z = r(\cos\theta + i\sin\theta)$

$\Rightarrow z^2 = r^2(\cos 2\theta + i\sin 2\theta)$

$\Rightarrow z^3 = r^3(\cos 3\theta + i\sin 3\theta)$

$\Rightarrow z^4 = r^4(\cos 4\theta + i\sin 4\theta)$

Prop. 1.3.6 For a positive integer n and a complex number $z = \cos\theta + i\sin\theta$,
$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Example Using de Moivre's Identity, show that the following trig. identity holds:

$$\cos(4\theta) = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$$

Solution Notice that

$$\cos 4\theta = \mathrm{Re}\left(\cos 4\theta + i\sin 4\theta\right)$$
$$= \mathrm{Re}\left((\cos\theta + i\sin\theta)^4\right)$$
$$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta. \quad \square$$

Roots of complex numbers

Def. 1.3.9 Let $\omega \neq 0$ and $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$. A number $z \in \mathbb{C}$ is an nth root of ω if $z^n = \omega$.

Let $z = r(\cos\theta + i\sin\theta)$

and $\omega = \rho(\cos\phi + i\sin\phi)$

So, $z^n = \omega$ becomes

$$r^n(\cos n\theta + i\sin n\theta) = \rho(\cos\phi + i\sin\phi).$$

$$\Leftrightarrow r^n = \rho \quad \text{and} \quad n\theta = \phi + 2k\pi$$

$k \in \mathbb{Z} = \{\ldots, -2, -1, \ldots\}$

$$\Leftrightarrow r = \sqrt[n]{\rho} \quad \text{and} \quad \theta = \frac{\phi + 2k\pi}{n}$$

with $k = 0, 1, \ldots, n-1$.

Prop. 1.3.10 Let $\omega = \rho(\cos\phi + i\sin\phi)$, $\omega \neq 0$.

The nth roots of ω are

$$z_{k+1} = \sqrt[n]{\rho}\left(\cos\left(\frac{\phi + 2k\pi}{n}\right) + i\sin\left(\frac{\phi + 2k\pi}{n}\right)\right)$$

with $k = 0, 1, 2, \ldots, n-1$.

Remarks

- The unique number z s.t. $z^n = \omega$, $\mathrm{Arg}(z) = \frac{\mathrm{Arg}(\omega)}{n}$ is the Principal nth root of ω.

- The Principal nth root is denoted by $\sqrt[n]{\omega}$.
- For all the nth roots, we use the notation $(\omega)^{1/n}$.

Example Find all the roots of

$$z^2 + z + 1 - i = 0$$

Solution. Quadratic Formula:

$$z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{-3+4i}}{2}$$

$|-3+4i| = 5$ and $\mathrm{Arg}(-3+4i) \cong 2.2142 = A$

$$\Rightarrow \sqrt{-3+4i} = \sqrt{5}\left(\cos(A/2) + i\sin(A/2)\right)$$

$$= 1 + 2i$$

Hences

$$z_1 = [-1 + (1+2i)]/2 = \boxed{i}$$

$$z_2 = [-1 - (1+2i)]/2 = \boxed{-1-i}$$