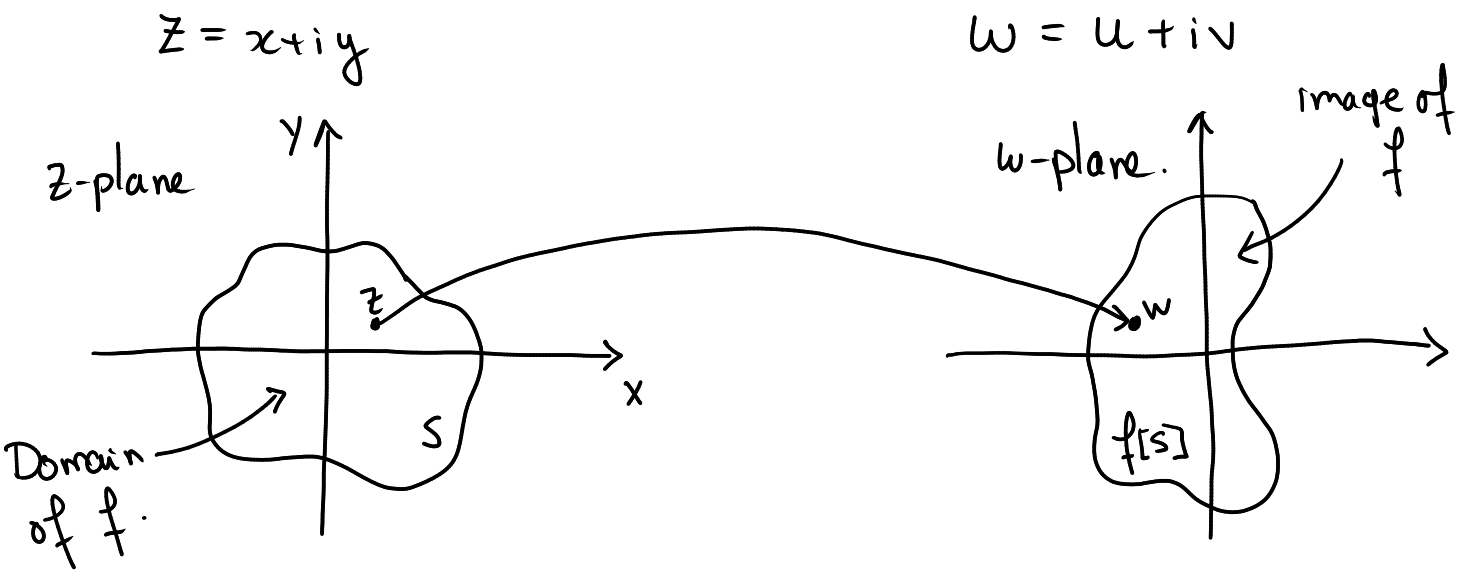
SECTION 1.4: Complex functions.

DEF. A **complex-valued function** f is a relation that assign $z \in S \subseteq \mathbb{C}$ to a $w = f(z) \in \mathbb{C}$.



Examples Link to Desmos: https://www.desmos.com/calculator/lanazhdjvb

① Affine transformations: $f(z) = az + b$, $a, b \in \mathbb{C}$

- $f(z) = z + b$, $b \in \mathbb{C}$, then f is a **translation**.
- $f(z) = az$, $a \in \mathbb{C}$:
  - $|a| = 1 \Rightarrow f$ is a **rotation**. ($a = \cos\theta + i\sin\theta$).
  - $a \in (0, \infty) \Rightarrow f$ is a **dilation**.

(2) Linear Fractional Transformation: $f(z) = \frac{az+b}{cz+d}$

with $ad - bc \neq 0$.

- $f(z) = \frac{1}{z}$, $z \neq 0$ (Inversion).

Real and Imaginary parts of a function

Since $w = f(z) \in \mathbb{C}$, we can write

$$w = f(z) = u(x,y) + i\,v(x,y)$$

where $u = \mathrm{Re}\, f$ is called the real part of f.

and $v = \mathrm{Im}\, f$ is called the imaginary part of f.

Example: $f(z) = z^2$, $z \in \mathbb{C}$.

Write $z = x + iy$

$$\Rightarrow \quad z^2 = x^2 - y^2 + i(2xy).$$

$$\Rightarrow \quad \mathrm{Re}\, f(z) = u(x,y) = x^2 - y^2.$$

$$\& \quad \mathrm{Im}\, f(z) = v(x,y) = 2xy.$$

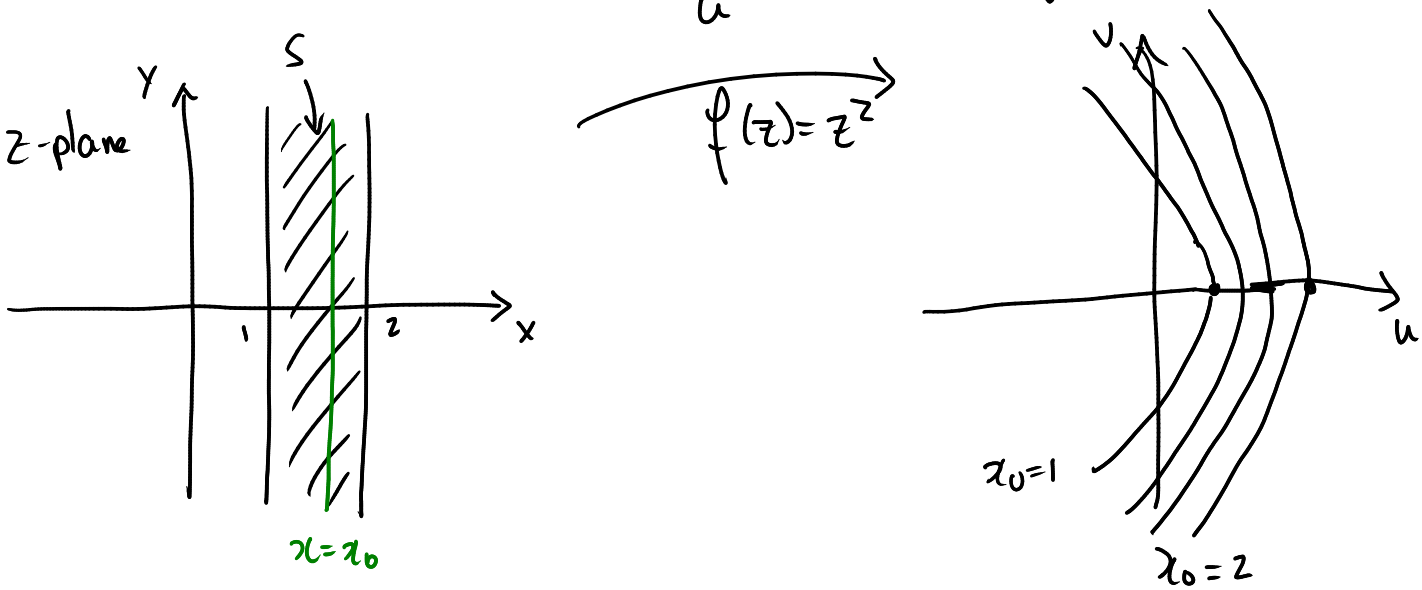
Example 1.4.4 $S = \{ z = x+iy : \ 1 \le x \le 2 \}$.

Find $f[S]$ under $f(z) = z^2$.

https://www.desmos.com/calculator/bl0hf6pgcg

Solution: We know that

$$f(z) = z^2 = \underbrace{x^2 - y^2}_{u} + i \underbrace{(2xy)}_{v}$$



Fix $x = x_0$, with $1 \le x_0 \le 2$.

$$\Rightarrow \quad u = x_0^2 - y^2 \quad \text{and} \quad v = 2x_0 y$$

We see that $y = \dfrac{v}{2x_0}$

$$\Rightarrow \quad \boxed{u = x_0^2 - \frac{v^2}{4x_0^2}}$$ ← parabola

$x_0 = 1 \Rightarrow \quad u = 1 - \dfrac{v^2}{4}$, $v \in \mathbb{R}$

$x_0 = 2 \Rightarrow \quad u = 4 - v^2/16$, $v \in \mathbb{R}$

Mapping in Polar Form

Writing $z = r\cos\theta + i r\sin\theta$ and

$$w = \rho(\cos\phi + i\sin\phi)$$

$$\Rightarrow \quad \rho = \rho(r,\theta) = |w| = |f(r\cos\theta + ir\sin\theta)|$$

$$\& \quad \phi = \phi(r,\theta) = \operatorname{Arg}\left[f(r\cos\theta + ir\sin\theta)\right]$$

Example 1.4.6 $S = \left\{ z : |z| \le \frac{3}{2},\ 0 \le \operatorname{Arg} z \le \frac{\pi}{4} \right\}$.

Find $f[S]$ under $f(z) = z^3$.

Solution. Write $z = r(\cos\theta + i\sin\theta)$

$$\Rightarrow \quad z^3 = r^3\cos(3\theta) + i r^3 \sin(3\theta)$$

$$\Rightarrow \quad \rho = \rho(r,\theta) = r^3$$

$$\& \quad \phi = \phi(r,\theta) = 3\theta$$