SECTION 1.8: Logarithms and powers

Log function(s)

Let $z \in \mathbb{C}$ and $w \in \mathbb{C}$.

$$w = \log(z) \iff e^{w} = z$$

Let $w = u + iv$ and $z = re^{i\theta}$ with $z \neq 0$. Then

$$e^{w} = z \iff e^{u} e^{iv} = r e^{i\theta}$$

$$\iff e^{u} = r \quad \text{and} \quad v = \theta + 2k\pi, \ k \in \mathbb{Z}$$

$$\iff u = \log(r) \quad \text{and} \quad v = \theta + 2k\pi, \ k \in \mathbb{Z}$$

Thus, the complex logarithm of $z \in \mathbb{C} \setminus \{0\}$ with $z = re^{i\theta}$ is

$$\log(z) = \log(r) + i(\theta + 2k\pi)$$

with $k \in \mathbb{Z}$.

Another notation:

$$\log(z) = \log|z| + i\arg(z)$$

$$= \{ \log|z| + (\mathrm{Arg}(z) + 2k\pi)i : k \in \mathbb{Z} \}$$

Example 1.8.1

(a) $\log(i) = \log|i| + (\mathrm{Arg}(i) + 2k\pi)i$

Here, $|i| = 1$

and $\mathrm{Arg}(i) = \pi/2$

$$\Rightarrow \log(i) = \log(1) + \left(\frac{\pi}{2} + 2k\pi\right)i$$

with $k \in \mathbb{Z}$.

(b) $\log(1+i) = \log\sqrt{2} + \left(\frac{\pi}{4} + 2k\pi\right)i$

with $k \in \mathbb{Z}$.

(c) $\log(-2) = \log(2) + (\pi + 2k\pi)i$

with $k \in \mathbb{Z}$.

$$\Rightarrow \log(-2) = \{ \ldots, \log 2 - 3\pi i, \log 2 - \pi i, \log 2 + \pi i, \ldots \}.$$

<u>DEF 1.8.2</u> The principal value or principal branch of the complex logarithm is defined by

$$\operatorname{Log}(z) = \overset{\ln}{\log}|z| + i \operatorname{Arg}(z)$$

for $z \neq 0$.

<u>Example 1.8.3</u>

(a) $\operatorname{Log}(i) = \log(1) + \frac{\pi}{2} i = i\frac{\pi}{2}$.

(b) $\operatorname{Log}(5) = \log(5)$

(c) $\operatorname{Log}(\underbrace{e^{6\pi i}}_{=1}) = \log(1) + i\,0 = 0$

Remarks

(1) $x \in \mathbb{R}$ and $x > 0 \Rightarrow \operatorname{Log}(x) = \log(x)$.

(2) $x \in \mathbb{R}$ and $x < 0 \Rightarrow \operatorname{Log}(x) = \log|x| + i\pi$

$\log|z| + i\operatorname{Arg}(z)$

(3) $\forall z \in \mathbb{C} \setminus \{0\}, \quad e^{\operatorname{Log} z} = z$.

But, $\operatorname{Log}(e^z)$ is not necessarily equal to z! In fact,

$$\operatorname{Log}(e^z) = z \iff -\pi < \operatorname{Im} z \le \pi.$$

(4) $x_1, x_2 \in \mathbb{R}$ and $x_1 > 0, \; x_2 > 0$

$$\Rightarrow \log(x_1 x_2) = \log(x_1) + \log(x_2).$$

But,

$$\operatorname{Log}((-1)(-1)) = \operatorname{Log}(1) = 0$$

and

$$\mathrm{Log}(-1) = i\pi \quad , \text{ so that}$$

$$\mathrm{Log}(-1) + \mathrm{Log}(-1) = 2\pi i \neq 0 = \mathrm{Log}((-1)(-1)).$$

Powers of z

For $x > 0$, and $a > 0$, then

$$x^a = e^{a \ln x}$$

For $z \in \mathbb{C} \setminus \{0\}$, and $a \in \mathbb{C} \setminus \{0\}$,

we define

$$z^a = e^{a \log z}.$$

Principle value of z^a:

$$z^a = e^{a \mathrm{Log} z} \quad , \quad z \neq 0.$$

<u>Example 1.8.7</u> Compute $(-i)^{1+i}$.

By the formula,

$$(-i)^{1+i} = e^{(1+i)\log(-i)}$$

1) $\log(-i) = \left\{ -\frac{\pi}{2} i + 2k\pi i : k \in \mathbb{Z} \right\}$

2) $(1+i)\log(-i) = \left\{ \frac{\pi}{2} - 2k\pi + \left(\frac{-\pi}{2} i + 2k\pi i\right) : k \in \mathbb{Z} \right\}$

$$= \left\{ \frac{\pi}{2} + 2k\pi - \frac{\pi}{2} i + 2k\pi i : k \in \mathbb{Z} \right\}.$$

3) $(-i)^{1+i} = \left\{ e^{\pi/2 + 2k\pi - \frac{\pi}{2} i + 2k\pi i} : k \in \mathbb{Z} \right\}$

$$= \left\{ e^{\pi/2 + 2k\pi} e^{-\pi/2 i + 2k\pi i} : k \in \mathbb{Z} \right\}$$

$$= \left\{ e^{\pi/2 + 2k\pi} e^{-\pi/2 i} : k \in \mathbb{Z} \right\}$$

$$= \left\{ -i e^{\pi/2 + 2k\pi} : k \in \mathbb{Z} \right\}.$$

Assume $a \in \mathbb{N}$. We have

$$z^a = e^{a \log z} = e^{a \operatorname{Log} z + 2k\pi a i}$$

for some $k \in \mathbb{Z}$. Since a is an integer

$$e^{a \operatorname{Log} z + 2k\pi a i} = e^{a \operatorname{Log} z} e^{2k\pi a i}$$
$$= e^{a \operatorname{Log} z}$$

Here, z^a has only one value which was expected when $a \in \mathbb{N}$.

1. If $a = \frac{p}{q}$, with $q \in \mathbb{N}$, $p \in \mathbb{Z}$.
   In this case, z^a has q distinct values.
2. $a \in \mathbb{C} \setminus \mathbb{Q}$, then z^a has ∞ many distinct values.