SECTION 2.2: Limits and Continuity

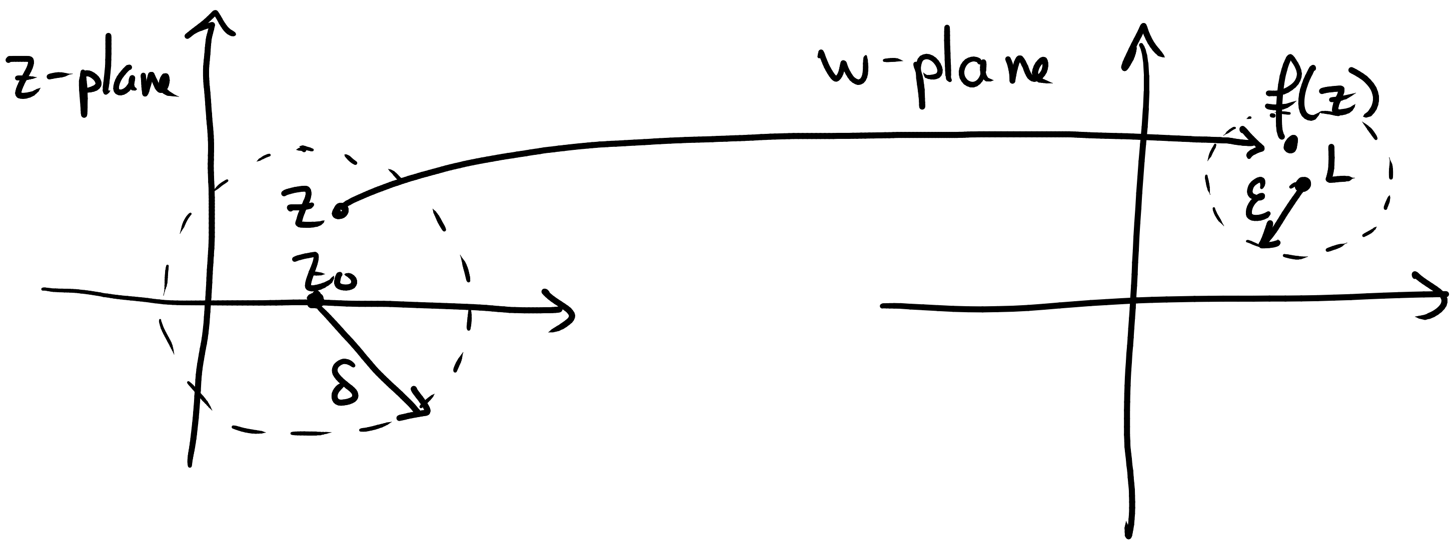
Limits

Def. Let $f: U \to \mathbb{C}$ where U is an open set.

We say L is the limit of f at $z_0 \in U$ if as z approaches z_0, $f(z)$ approaches L, that is

$\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$$0 < |z - z_0| < \delta \Rightarrow |f(z) - L| < \varepsilon .$$



Prop. 2.2.2 (Uniqueness of limits)
If $f: U \to \mathbb{C}$ is a function defined on an open set U, and if f has limit L at $z_0 \in U$, then L is unique.

Proof. Assume that there are two limits L_1, L_2 at z_0 with $L_1 \neq L_2$. Then

$$|L_1 - L_2| \neq 0$$

Let $\varepsilon = \frac{|L_1 - L_2|}{4} > 0$. By def of limits, $\exists \delta_1 > 0$ and $\exists \delta_2 > 0$ such that

$$0 < |z - z_0| < \delta_1 \Rightarrow |f(z) - L_1| < \frac{|L_1 - L_2|}{4}$$

and

$$0<|z-z_0|<\delta_2 \Rightarrow |f(z)-L_2|<\frac{|L_1-L_2|}{4}$$

So, if $|z-z_0|<\min\{\delta_1,\delta_2\}$, then

$$\begin{aligned}|L_1-L_2| &= |L_1-f(z)+f(z)-L_2|\\ &\leq |L_1-f(z)|+|f(z)-L_2|\\ &< \frac{|L_1-L_2|}{4}+\frac{|L_1-L_2|}{4}\\ &= \frac{|L_1-L_2|}{2}\end{aligned}$$

$$\Rightarrow |L_1-L_2|<\frac{|L_1-L_2|}{2}$$

contradiction!

So, $L_1=L_2$. $\square$

Notation:

$$L=\lim_{z\to z_0} f(z) \quad \text{or} \quad f(z)\longrightarrow L \ (z\to z_0).$$

Thm. 2.2.9 Let $U \subset \mathbb{C}$ be an open set. Let $f: U \to \mathbb{C}$ be a function with

$$f(z) = u(z) + iv(z), \quad z \in U.$$

Then

$$\lim_{z \to z_0} f(z) = a + ib \iff \begin{cases} \lim_{z \to z_0} u(z) = a \\ \lim_{z \to z_0} v(z) = b \end{cases}$$

Proof.

($\Rightarrow$) Assume $\lim_{z \to z_0} f(z) = L = a + ib$.

WTS: $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$0 < |z - z_0| < \delta \Rightarrow |u(z) - a| < \varepsilon.$$

and

$\forall \varepsilon > 0$, $\exists \delta > 0$, such that

$$0 < |z - z_0| < \delta \Rightarrow |v(z) - b| < \varepsilon.$$

Let $\varepsilon > 0$. By def. of $\lim_{z \to z_0} f(z) = L$,

$\exists \delta > 0$ s.t.

$0 < |z - z_0| < \delta \Rightarrow |f(z) - L| < \varepsilon$. $(*)$

<u>Recall</u>:

$$|\mathrm{Re}\, w| \leq |w|.$$

Let $z \in U$ such that $0 < |z - z_0| < \delta$.

then

$$|u(z) - a| = |\mathrm{Re}(f(z) - \underbrace{(a+ib)}_{=L})|$$
$$\leq |f(z) - L|$$
$$< \varepsilon.$$

<u>Summary</u>: we found a $\delta > 0$ s.t.

$0 < |z - z_0| < \delta \Rightarrow |u(z) - a| < \varepsilon$.

Repeat same argument for $v(z)$.

($\Leftarrow$) Assume $\lim_{z \to z_0} u(z) = a$ and $\lim_{z \to z_0} v(z) = b$.

WST $\lim_{z \to z_0} f(z) = a + ib$.

i.e. $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$0 < |z - z_0| < \delta \Rightarrow |f(z) - (a+ib)| < \varepsilon$.

Let $\varepsilon > 0$. From the definition of limits, $\exists \delta_1 > 0$, $\exists \delta_2 > 0$ such that

$0 < |z - z_0| < \delta_1 \Rightarrow |u(z) - a| < \varepsilon/2$ (Δ)

& $0 < |z - z_0| < \delta_2 \Rightarrow |v(z) - b| < \varepsilon/2$. $(\circ)$

Recall:

$$|w| \le |\operatorname{Re} w| + |\operatorname{Im} w|$$

Let $\delta := \min\{\delta_1, \delta_2\}$.

If $|z-z_0|<\delta$, then

$$|f(z)-(a+ib)| \le |u(z)-a|+|v(z)-b|$$
$$< \varepsilon/2 + \varepsilon/2$$
$$= \varepsilon$$

<u>Summary</u>: $\forall \varepsilon>0$, $\exists \delta>0$ s.t.

$$0<|z-z_0|<\delta \Rightarrow |f(z)-(a+ib)|<\varepsilon.$$

So, $\lim_{z\to z_0} f(z) = a+ib$. $\square$

<u>Example</u> Compute

(a) $\lim_{z\to z_0} z$ (b) $\lim_{z\to z_0} z^2$.

<u>Solution</u>.

(a) $z\to z_0 \Leftrightarrow x\to x_0$ and $y\to y_0$.

(b) $z^2 = (x+iy)(x+iy)$

$$= \underbrace{x^2 - y^2}_{u(z)} + \underbrace{(2xy)}_{v(z)} i$$

$$\lim_{z \to z_0} (x^2 - y^2) = \lim_{(x,y) \to (x_0,y_0)} x^2 - y^2$$

$$= x_0^2 - y_0^2 \quad \text{((Calculus).}$$

$$\lim_{z \to z_0} 2xy = \lim_{(x,y) \to (x_0,y_0)} 2xy$$

$$= 2x_0 y_0$$

Hence:

$$\lim_{z \to z_0} z^2 = \overbrace{(x_0^2 - y_0^2)}^{a} + \overbrace{(2x_0y_0)}^{b} i$$

$$= z_0^2$$

In fact:

$$\lim_{z \to z_0} z^n = z_0^n \, .$$

Properties of limits

① $$\lim_{z\to z_0}\left(f(z)+g(z)\right)=\lim_{z\to z_0}f(z)+\lim_{z\to z_0}g(z)$$

② $$\lim_{z\to z_0}f(z)g(z)=\left(\lim_{z\to z_0}f(z)\right)\left(\lim_{z\to z_0}g(z)\right)$$

③ $$\lim_{z\to z_0}\frac{f(z)}{g(z)}=\frac{\lim_{z\to z_0}f(z)}{\lim_{z\to z_0}g(z)}$$

if $\lim_{z\to z_0}g(z)\neq 0$.

④ $$\lim_{z\to z_0}|z|=|z_0|$$

⑤ $$\lim_{z\to z_0}\overline{z}=\overline{\lim_{z\to z_0}z}.$$

THM (Squeeze Theorem; Thm 2.2.5)

Let f, g be defined on an open set $U \subset \mathbb{C}$ and let $z_0 \in U$.

a) $\lim_{z \to z_0} f(z) = 0$ and $|g(z)| \leq |f(z)|$ in a deleted neighborhood of z_0, then $\lim_{z \to z_0} g(z) = 0$

b) $\lim_{z \to z_0} f(z) = 0$ and g is bounded in a deleted neighborhood of z_0, then $\lim_{z \to z_0} f(z)g(z) = 0$.

Remark: g is bounded in a deleted neighborhood of z_0 means $\exists r > 0$ and $\exists M > 0$ s.t.

$$|g(z)| \leq M \quad , \forall z \in B_r'(z_0)$$

Proof.

(a) Assume $\lim_{z \to z_0} f(z) = 0$ and

$|g(z)| \le |f(z)|$, for all $z \in B_r'(z_0)$.

Let $\varepsilon > 0$. From the assumption,

$\exists \delta_1 > 0$ s.t.

$$0 < |z - z_0| < \delta_1 \Rightarrow |f(z)| < \varepsilon .$$

Let $\delta = \min\{\delta_1, r\}$. If $0 < |z - z_0| < \delta$, then

$$|g(z)| \le |f(z)| < \varepsilon .$$

So, $\lim_{z \to z_0} g(z) = 0$.

(b) From assumption, $\exists r > 0$, $\exists M > 0$ s.t.

$$|g(z)| \le M , \quad \forall z \in B_r'(z_0)$$

Now, for $z \in B_r'(z_0)$

$$|f(z)g(z)| \leq |f(z)||g(z)|$$
$$\leq M|f(z)|$$

for any $z \in B_r'(z_0)$.

Also,

$$\lim_{z \to z_0} Mf(z) = M \lim_{z \to z_0} f(z)$$
$$= M \cdot 0 = 0.$$

From part (a) applied to $f(z)g(z)$:

$$\lim_{z \to z_0} f(z)g(z) = 0. \quad \square$$

Limits at infinity

We assume f is at least defined in a neighborhood of $z_0 \in \mathbb{C}$

or on $\{z \in \mathbb{C} : |z| > R\}$.

① $\lim_{z \to z_0} f(z) = \infty$ if $\forall M > 0, \exists \delta > 0$

$0 < |z - z_0| < \delta \Rightarrow |f(z)| > M$.

② $\lim_{z \to \infty} f(z) = L$ if $\forall \varepsilon > 0, \exists R > 0$

$|z| > R \Rightarrow |f(z) - L| < \varepsilon$.

③ $\lim_{z \to \infty} f(z) = \infty$ if $\forall M > 0, \exists R > 0$

$|z| > R \Rightarrow |f(z)| > M$.

<u>Remark:</u>

$$\lim_{z \to \infty} f(z) = L \Leftrightarrow \lim_{z \to 0} f(1/z) = L.$$

Using this trick,

$$\lim_{z\to\infty} \frac{1}{z^n} = \lim_{z\to 0} \frac{1}{(1/z)^n} = \lim_{z\to 0} z^n = 0.$$

Continuous Functions

DEF (modif. of Def. 2.2.12)

A function f is continuous at z_0 if

(a) f is defined in a neighborhood of z_0.

(b) $\lim_{z\to z_0} f(z) = f(z_0)$.

A function f defined on an open set U is continuous on U if it is continuous $\forall z_0 \in U$.

Consequences:

① f, g continuous at z_0, then $af + bg$ is continuous at z_0 fn any $a, b \in \mathbb{C}$ and fg is continuous at z_0.

② f, g are continuous at z_0, then $\frac{f}{g}$ is continuous at z_0 provided that $g(z_0) \neq 0$.

③ Any polynomial $p(z) = a_n z^n + \cdots + a_0$ is continuous on $\mathbb{C}$.

④ Any rational function

$$f(z) = \frac{p(z)}{q(z)} = \frac{a_n z^n + \cdots + a_0}{b_m z^m + \cdots + b_0}$$

is continuous on $\mathbb{C} \setminus \{z : q(z) = 0\}$.

⑤ The function $f(z) = \bar{z}$ is continuous on $\mathbb{C}$.

<u>Thm</u> Let f be continuous at z_0 and h be continuous at $f(z_0)$.

(a) $h \circ f$ is continuous at z_0.

(b) $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are continuous at z_0

(c) The function $g(z) = |f(z)|$ is continuous at z_0.

<u>Proof.</u> (a) is proved basically in the same way as in Calculus or 331.

(b) $\mathrm{Re} f(z) = \dfrac{f(z) + \overline{f(z)}}{2}$

from this and the properties of

continuous functions, Re f is continuous. Similarly,

$$\operatorname{Im} f(z) = \frac{f(z) - \overline{f(z)}}{2i} .$$

(c) Here, we have

$$g(z) = |f(z)|^2 = f(z)\,\overline{f(z)}$$

Product and composition of cont. fcts are continuous, so g is cont. So, we know $\sqrt{\cdot}$ is continuous for real-numbers, so $\sqrt{|f(z)|^2} = |f(z)|$ is continuous. $\square$

Thm 2.2.21

The principal branch $\mathrm{Log}(z)$ is continuous on $\mathbb{C} \setminus (-\infty, 0]$.

Proof. For $z \neq 0$,

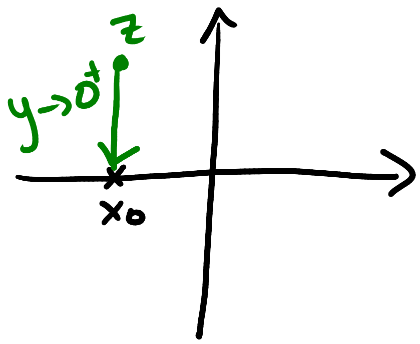
$$\mathrm{Log}(z) = \log|z| + i\,\mathrm{Arg}(z).$$

Recall: $-\pi < \mathrm{Arg}(z) \leq \pi$.

$\mathrm{Log}(z)$ is not defined at $z=0$, so discontinuous at $z=0$.

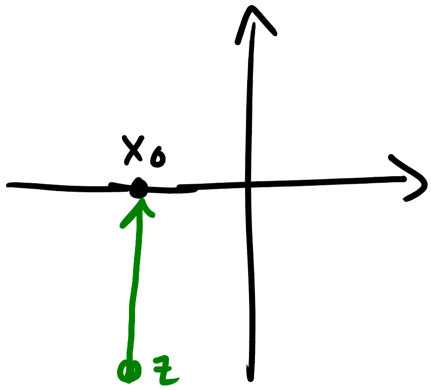
① Let $z_0 = x_0 \in (-\infty, 0)$.

Let $z = x_0 + iy$, $y \in \mathbb{R}$.



$$\lim_{z \to z_0} \mathrm{Log}(z) = \lim_{y \to 0^+} \log|z| + i\,\mathrm{Arg}(z)$$

$$= \log|x_0| + i\pi$$



$$\lim_{z \to z_0} \mathrm{Log}(z) = \lim_{y \to 0^-} \log|z| + i\,\mathrm{Arg}(z)$$

$$= \log|x_0| + i(-\pi)$$

$$= \log|x_0| - i\pi$$

Here, there seems to have two possible limits, impossible!

(2) Let $z_0 = x_0 + i y_0$ not in $(-\infty, 0]$.

Let z be a point in $\mathbb{C} \setminus \{0\}$.

Let $\theta_0 = \mathrm{Arg}(z_0)$ and

$\theta = \mathrm{Arg}(z)$.

<u>Goal:</u> $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$$|z - z_0| < \delta \Rightarrow |\theta - \theta_0| < \varepsilon.$$

From Desmos https://www.desmos.com/calculator/mvhdxcjr4d

we see that

$$|\theta - \theta_0| \leq 2\varphi$$

where $\omega_1 = e^{i(\theta_0 - \varphi)}$, $\omega_2 = e^{i(\theta_0 + \varphi)}$

(see on Desmos). Now, we have

$2\varphi \leq 2\pi\delta$. Set $\delta = \dfrac{\varepsilon}{(2\pi+1)}$. Then

$$|z - z_0| < \delta \Rightarrow |\theta - \theta_0| \leq 2\pi\delta$$

$$\Rightarrow |\theta - \theta_0| < \varepsilon.$$

So, $\mathrm{Arg}(z)$ is continuous.

Also, $\log|z|$ is continuous.

So, $\mathrm{Log}(z) = \log|z| + i\,\mathrm{Arg}(z)$

is continuous. $\square$

Remarks:

1. $\mathrm{Arg}(z)$ is continuous on $\mathbb{C} \setminus (-\infty, 0]$
2. z^{α} is cont. on $\mathbb{C} \setminus (-\infty, 0]$, $\alpha \neq 0$.

(3) Recall that e^z is continuous because

$$e^z = e^x \cos y + i e^x \sin y.$$