M444 – Complex Analysis

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Chapter 2

Section 2.3: Complex Derivative

Definition (Complex Derivative ; Definition 2.3.1)

Let f be a function defined on an open set $U \subset \mathbb{C}$ and let $z_0 \in U$. We say that f has a **complex derivative** at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{exists.}$$

Remarks :

1. The limit is called the **complex derivative** at z_0 and is denoted by $f'(z_0)$.
2. f is **analytic** on U if it has a complex derivative at every $z_0 \in U$.
3. f is **analytic at a point w in U** if it is analytic on some neighborhood of w contained in U.

Here are some examples of functions that are analytic everywhere.

1. $f(z) = az + b$. Indeed, we have

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{a(z - z_0)}{z - z_0} = a.$$

   Hence, $f'(z_0) = a$.

2. $f(z) = z^2$.Indeed, we have

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0} = \lim_{z \to z_0} (z + z_0) = 2z_0.$$

   Hence[1], $f'(z_0) = 2z_0$.

Definition

A function is called **entire** if it is analytic on $\mathbb{C}$.

1. In general, z^n is analytic on $\mathbb{C}$ and $(z^n)' = nz^{n-1}$. See homework 2.

Here is a non-example : $f(z) = \overline{z}$.

Fix $z_0 = x_0 + iy_0 \in \mathbb{C}$.

1. Let $z = x + iy_0$, with $x \to x_0$. Then

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{x \to x_0} \frac{x - iy_0 - x_0 + iy_0}{x - x_0} = \lim_{x \to x_0} \frac{x - x_0}{x - x_0} = 1.$$

2. But, letting $z = x_0 + iy$, with $y \to y_0$. Then

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{y \to y_0} \frac{x_0 - iy - x_0 + iy_0}{i(y - y_0)} = -\lim_{y \to y_0} \frac{y - y_0}{y - y_0} = -1$$

Therefore, $f'(z_0)$ does not exist.

Questions : What about $\operatorname{Re} f$ and $\operatorname{Im} f$?

For any functions f and g analytic on an open set U :

① Any analytic function is continuous.

② $c_1 f + c_2 g$ is analytic on U and

$$(c_1 f + c_2 g)'(z) = c_1 f'(z) + c_2 g'(z) \quad (\forall z \in U).$$

③ fg is analytic on U and

$$(fg)'(z) = f'(z)g(z) + f(z)g'(z) \quad (\forall z \in U).$$

Examples : polynomials.

④ f/g is analytic on $W := U \backslash \{z : g(z) = 0\}$ and

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2} \quad (\forall z \in W).$$

⑤ If g is analytic on U and f is analytic on V containing $g(U)$, then $f \circ g$ is analytic and

$$(f \circ g)'(z) = f'(g(z))g'(z) \quad (\forall z \in U).$$

Examples : rational functions are analytic on their domain.

Theorem (Reverse Chain Rule ; Theorem 2.3.12)

Assumptions :

1. g *is continuous on an open set* V *;*
2. f *is analytic on an open set* U *such that* $g(V) \subset U$ *;*
3. $h = f \circ g$ *is analytic on* V *;*
4. $f'(g(z)) \neq 0$ *for any* $z \in V$*.*

Then g *is analytic on* V *and*

$$g'(z) = \frac{h'(z)}{f'(g(z))} \quad z \in V.$$

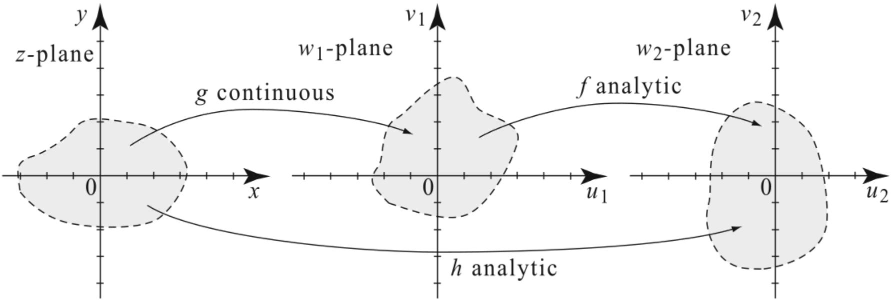

Fig. 2.14 In the reverse chain rule we suppose that g is continuous and that $h = f \circ g$ is analytic and we conclude that g is analytic.

Consider $g(z) = \sqrt[n]{z}$, for $z \in \mathbb{C} \backslash \{0\}$.

① Notice that

$$g(z) = e^{\frac{1}{n}\operatorname{Log}(z)}$$

so g is continuous on $\mathbb{C} \backslash (-\infty, 0]$.

② Set $f(z) = z^n$ so that $h(z) = f(g(z)) = (\sqrt[n]{z})^n = z$.

③ Both f and h are analytic.

④ From the Inverse Chain Rule, g is analytic on $\mathbb{C} \backslash (-\infty, 0]$ and

$$g'(z) = \frac{h'(z)}{f'(g(z))} = \frac{1}{n(\sqrt[n]{z})^{n-1}} = \tfrac{1}{n}(\sqrt[n]{z})^{1-n}$$

for $z \in \mathbb{C} \backslash (-\infty, 0]$.

⑤ Notice that

$$g'(z) = \tfrac{1}{n}(\sqrt[n]{z})^{1-n} = \tfrac{1}{n}(e^{\frac{1}{n}\operatorname{Log} z})^{1-n} = \tfrac{1}{n}e^{\frac{1-n}{n}\operatorname{Log} z} = \tfrac{1}{n}z^{\frac{1-n}{n}}.$$