M444 – Complex Analysis

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Chapter 3

Section 3.1: Paths (Contours) in the Complex Plane

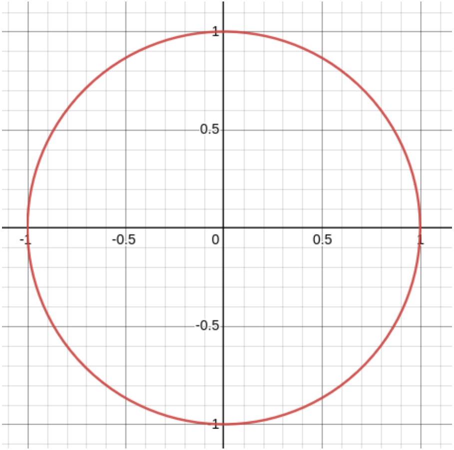

Figure – Circle $x^2 + y^2 = 1$

① From Calculus :

$$\begin{cases} x(t) = \cos(t) \\ y(t) = \sin(t) \end{cases}$$

$0 \leq t \leq 2\pi.$

② Complex numbers :

$$\begin{aligned} z(t) &= x(t) + iy(t) \\ &= \cos(t) + i\sin(t) \\ &= e^{it} \end{aligned}$$

$0 \leq t \leq 2\pi.$

Definition

A **parametric form** of a curve in the complex plane is a function $z : [a, b] \to \mathbb{C}$ where $z(t) = x(t) + iy(t)$.

Remarks :

1. In the textbook (see Definition 3.1.1), the authors also use the notation $\gamma(t)$ to represent a parametric form of a curve.
2. Representation of a curve : `https://www.desmos.com/calculator/ardibjhbww`.
3. Here, $z(a)$ is the **initial point** and $z(b)$ is the **terminal point**.
4. If $z(a) = z(b)$, the curve is said to be **closed**.
5. If $z(t_1) \neq z(t_2)$ for any $t_1 \neq t_2$, then the curve is said to be **simple**.

Useful examples :

1. A **directed line segment** from z_1 to z_2 :

$$z(t) = (1 - t)z_1 + tz_2$$

for $0 \leq t \leq 1$. We use the notation $[z_1, z_2]$.

2. A **circle** with center z_0 and radius R :

$$z(t) = z_0 + Re^{it}$$

for $0 \leq t \leq 2\pi$.

3. An **epicycloid** (see #26 in problem set) :

$$z(t) = (a + b)e^{it} - be^{\frac{a+b}{b}it}.$$

Visualization : `https://www.desmos.com/calculator/hqvgaimgtr`

GOAL : Given a parametrization $z : [a, b] \to \mathbb{C}$ of a curve, get another parametrization $w : [a, b] \to \mathbb{C}$ starting at $z(b)$ and ending at $z(a)$.

1. Notice that to start the parameter t at b, we can map the parameter a to b using
$$t \mapsto b + a - t$$
2. Therefore, setting
$$w(t) = z(b + a - t)$$
for $a \leq t \leq b$ achieves what we want !
3. Click Desmos.

Definition

Given a parametrization $z(t)$ of a curve, the new parametrization $w(t) = z(b + a - t)$ is called the **reverse parametrization** of $z(t)$.

Let $z : (a, b) \to \mathbb{C}$ be a complex-valued function defined (a, b).

Since $z(t) \in \mathbb{C}$ there are two real-valued functions $x : (a, b) \to \mathbb{C}$ and $y : (a, b) \to \mathbb{C}$ such that

$$z(t) = x(t) + iy(t).$$

Definition

For a complex-valued function $z : (a, b) \to \mathbb{C}$, the derivative of z at t is defined as

$$\frac{dz}{dt}(t) = \frac{dx}{dt}(t) + i\frac{dy}{dt}(t)$$

if $\frac{dx}{dt}(t)$ and $\frac{dy}{dt}(t)$ exists at t.

Remarks :

1. We also denote the derivative $\frac{dz}{dt}(t)$ by $z'(t)$.
2. All the rules for differentiation still hold.

① Let $z(t) = (1+t) + t^2 i$. Then,

$$\frac{d}{dt}z(t) = \frac{d}{dt}(1+t) + \frac{d}{dt}(t^2)i = 1 + 2ti.$$

② Let $z(t) = \frac{1+t^2+i}{1-i+t}$. By the quotient rule

$$\begin{aligned} z'(t) &= \frac{(1+t^2+i)'(1-i+t) - (1+t^2+i)(1-i+t)'}{(1-i+t)^2} \\ &= \frac{(2t)(1-i+t) - (1+t^2+i)(1)}{(1-i+t)^2} \\ &= \frac{2t - i2t + 2t^2 - 1 - t^2 - i}{(1-i+t)^2} \\ &= \frac{-1+2t+t^2 - i(2t+1)}{(1-i+t)^2}. \end{aligned}$$

Example : Let $w(t) = (2 + i)) \cos(3it)$. Then, we see that

$$w(t) = F(z(t))$$

where $F(z) = (2 + i) \cos(z)$ and $z(t) = 3it$. Therefore

$$w'(t) = F'(z(t))z'(t) = -(2 + i) \sin(3it)(3i) = -(-3 + 6i) \sin(3it).$$

Theorem (Theorem 3.1.8)

① Assume that $z(t)$ is a differentiable complex-valued function on (a, b).

② Assume that F is an analytic function on an open set U containing all the values of $z(t)$.

③ Let $w(t) = F(z(t))$, for $a < t < b$.

Then w is differentiable on (a, b) and

$$w'(t) = F'(z(t))z'(t).$$

Example : The curve

$$z(t) = \begin{cases} 3t(1+i) & 0 \leq t \leq \frac{1}{3} \\ 3+i-6t & \frac{1}{3} \leq t \leq \frac{2}{3} \\ (-1+i)(3-3t) & \frac{2}{3} \leq t \leq 1 \end{cases}$$

has the following characteristics :

1. Continuous on $[0,1]$.
2. Differentiable and continuous everywhere, except at a finite number of points.

Definition

A **path** is a curve $z(t)$ defined on a closed interval $[a,b]$ which is **piecewise continuously differentiable**, that is

1. continuous on the interval of definition $[a,b]$.
2. differentiable everywhere, except at a finite number of points.
3. the derivative $z'(t)$ is continuous where it exists.

Definition

A **polygonal path** $[z_1, z_2, \dots, z_n]$ is the union of the directed segments $[z_1, z_2]$, $[z_2, z_3]$, ..., $[z_{n-1} z_n]$.

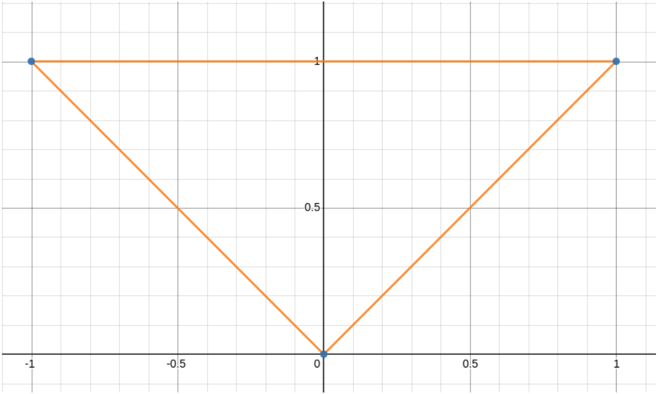

Figure – The polygonal curve from the previous example