M444 – Complex Analysis

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Chapter 3

Section 3.2: Complex Integration

Definition

Let $f : [a, b] \to \mathbb{C}$ be a continuous complex-valued function.

$$\int_a^b f(t)\,dt := \int_a^b \operatorname{Re} f(t)\,dt + i \int_a^b \operatorname{Im} f(t)\,dt.$$

Example. Consider the function $f(t) = t^2 + it$. Then

$$\int_1^3 f(t)\,dt = \int_1^3 t^2\,dt + i \int_1^3 t\,dt = \frac{t^3}{3}\Big|_1^3 + i\,\frac{t^2}{2}\Big|_1^3 = \frac{26}{3} + 4i.$$

Properties (Proposition 3.2.3) :

① Sum : $\int_a^b \alpha f(t) + \beta g(t)\,dt = \int_a^b f(t)\,dt + \beta \int_a^b g(t)\,dt.$

② Cut : $\int_a^b f(t)\,dt = \int_a^c f(t)\,dt + \int_c^b f(t)\,dt.$

③ By parts : $\int_a^b f(t)g'(t)\,dt = f(t)g(t)|_a^b - \int_a^b f'(t)g(t)\,dt.$

④ Abs. Value : $\left|\int_a^b f(t)\,dt\right| \leq \int_a^b |f(t)|\,dt.$

Definition

A function F is called an **antiderivative** of a continuous complex-valued function on (a, b) if

$$F'(t) = f(t).$$

Example. Let $f(t) = e^{3it}$, for $0 \leq t \leq 2\pi$. Then, $F(t) = \frac{1}{3i} e^{3it}$ is an antiderivative for $f(t)$ because

$$F'(t) = \tfrac{1}{3i}(e^{3it})' = \tfrac{1}{3i}(3ie^{3it}) = e^{3it} = X(t) + iY(t).$$

Now, we get

$$\begin{aligned}
\int_0^{2\pi} e^{3it}\,dt = \int_0^{2\pi} F'(t)\,dt &= \int_0^{2\pi} X'(t)\,dt + i\int_0^{2\pi} Y'(t)\,dt \\
&= X(t)\big|_0^{2\pi} + i\ Y(t)\big|_0^{2\pi} \\
&= F(t)\big|_0^{2\pi} \\
&= e^{3i(2\pi)} - e^{3i(0)} = 0.
\end{aligned}$$

Example. Consider the function

$$f(t) = \begin{cases} e^{i\pi t} & -1 \leq t \leq 0 \\ t & 0 < t \leq 1. \end{cases}$$

Then,

$$\begin{aligned} \int_{-1}^{1} f(t)\,dt &:= \int_{-1}^{0} f(t)\,dt + \int_{0}^{1} f(t)\,dt \\ &= \int_{-1}^{0} e^{i\pi t}\,dt + \int_{0}^{1} t\,dt \\ &= \frac{e^{i\pi t}}{i\pi}\bigg|_{-1}^{0} + \frac{t^2}{2}\bigg|_{0}^{1} \\ &= \left(\frac{e^{i\pi(0)} - e^{i\pi(-1)}}{i\pi}\right) + \left(\frac{(1)^2 - (0)^2}{2}\right) \\ &= \frac{2}{i\pi} + \frac{1}{2} \\ &= \frac{1}{2} - i\frac{2}{\pi}. \end{aligned}$$

Definition

1. Let $\gamma(t) = x(t) + iy(t)$ be a path and $\gamma := \gamma([a, b])$.
2. Let f be a continuous complex-valued function on an open set containing γ.

The **contour integral** of f over γ is

$$\int_\gamma f(z)\,dz := \int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

Example. Let $C = \{e^{it} : 0 \leq t \leq 2\pi\}$. Then, $\gamma(t) = e^{it}$ with $0 \leq t \leq 2\pi$. Let $f(z) = 1/z$.

$$\int_\gamma f(z)\,dz = \int_0^{2\pi} f(\gamma(t))\gamma'(t)\,dt = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\,dt = \int_0^{2\pi} i\,dt = 2\pi i.$$

Theorem (Proposition 3.2.12)

- Let $\gamma : [a, b] \to \mathbb{C}$ be a path and $\gamma := \gamma([a, b])$.
- Let $\gamma^* : [a, b] \to \mathbb{C}$ be the reverse path and $\gamma^* := \gamma^*([a, b])$.
- Let f, g be continuous functions on an open set containing C, the trace of γ (or γ^*).
- Let α, β be complex numbers.

Then

① $$\int_\gamma \alpha f(z) + \beta g(z)\, dz = \alpha \int_\gamma f(z)\, dz + \beta \int_\gamma g(z)\, dz.$$

② $$\int_{\gamma^*} f(z)\, dz = -\int_\gamma f(z)\, dz.$$

Example. Let $\gamma = \{e^{it} : 0 \leq t \leq 2\pi\}$ and let $f(z) = \operatorname{Re} z$. Then

$$\int_\gamma f(z)\,dz = \int_\gamma \operatorname{Re} z\,dz = \int_\gamma \frac{z+\overline{z}}{2}\,dz = \frac{1}{2}\int_\gamma z\,dz + \frac{1}{2}\int_\gamma \overline{z}\,dz.$$

With $\gamma(t) = e^{it}$ $(0 \leq t \leq 2\pi)$, we have

$$\int_\gamma z\,dz = \int_0^{2\pi} \gamma(t)\gamma'(t)\,dt = \int_0^{2\pi} e^{it} ie^{it}\,dt = i\int_0^{2\pi} e^{2it}\,dt = 0.$$

Also,

$$\int_\gamma \overline{z}\,dz = 2\pi i$$

Hence,

$$\int_\gamma f(z)\,dz = \frac{1}{2}(0) + \tfrac{1}{2}(2\pi i) = \pi i.$$

Definition

Let $\gamma : [a, b] \to \mathbb{C}$ be a parametrization of a curve. The length of the curve is given

$$\ell(\gamma) := \int_a^b \sqrt{|x'(t)|^2 + |y'(t)|^2}\, dt = \int_a^b |\gamma'(t)|\, dt.$$

Example : Consider $\gamma(t) = \frac{1}{5}t^5 + \frac{i}{4}t^4$, $0 \leq t \leq 1$. Then,

$$\gamma'(t) = t^4 + it^3 \quad \Rightarrow \quad |\gamma'(t)| = \sqrt{t^8 + t^6} = t^3\sqrt{t^2 + 1}.$$

Hence,

$$\ell(\gamma) = \int_0^1 |\gamma'(t)|\, dt = \int_0^1 t^3\sqrt{1 + t^2}\, dt = \tfrac{2}{15}(1 + \sqrt{2}) \approx 0.3219.$$

Theorem

1. Let $\gamma : [a, b] \to \mathbb{C}$ be a parametrization of a curve ;
2. Let f be a continuous function on an open set containing γ

If $|f(z)| \leq M$ for any $z \in \gamma$, then

$$\Big| \int_{\gamma} f(z)\, dz \Big| \leq M\ell(\gamma).$$

Proof : From the property of the integrals,

$$\Big| \int_{\gamma} f(t)\, dz \Big| = \Big| \int_a^b f(z(t))z'(t)\, dt \Big| \leq \int_a^b |f(z(t))z'(t)|\, dz.$$

Now, $|f(z(t))z'(t)| = |f(z(t))||z'(t)| \leq M|z'(t)|$ and so

$$\int_a^b |f(z(t))z'(t)|\, dt \leq \int_a^b M|z'(t)|\, dt = M \int_a^b |z'(t)|\, dt = M\ell(\gamma). \quad \square$$