M444 – Complex Analysis

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Chapter 3

Section 3.3: Independence of Path

Theorem (Theorem 3.3.4)

Let f be a continuous complex-valued function on a **region** Ω. Assume there is an analytic function F on Ω such that $f(z) = F'(z)$, $\forall z \in \Omega$. Then $\forall z_1, z_2 \in \Omega$ and any path $\gamma \subset \Omega$ joining z_1 to z_2, the integral

$$\int_\gamma f(z)\,dz = F(z_2) - F(z_1).$$

Proof. Recall that $\frac{d}{dt}(F(z(t))) = F'(z(t))z'(t)$.

Let $z : [a, b] \to \mathbb{C}$ be a parametrization of γ with $z(a) = z_1$ and $z(b) = z_2$. Then

$$\begin{aligned}\int_\gamma f(z)\,dz = \int_a^b F'(z(t))z'(t)\,dt &= \int_a^b \tfrac{d}{dt}(F(z(t)))\,dt \\ &= F(z(b)) - F(z(a))\end{aligned}$$

hence the result. □

Example. Consider

$$\int_{[z_1,z_2,z_3]} 3(z-1)^2\,dz$$

where $z_1 = 1$, $z_2 = i$, and $z_3 = 1+i$.

Consider $\Omega = \mathbb{C}$ and $F(z) = (z-1)^3$. Then F is analytic on Ω and $F'(z) = 3(z-1)^2$.

The path $[z_1, z_2, z_3] = [z_1, z_2] \cup [z_2, z_3] \subset \Omega$. Hence by Theorem 3.3.4

$$\begin{aligned}
\int_{[z_1,z_2,z_3]} 3(z-1)^2\,dz &= \int_{[z_1,z_2]} 3(z-1)^2\,dz + \int_{[z_2,z_3]} 3(z-1)^2\,dz \\
&= F(z_2) - F(z_1) + F(z_3) - F(z_2) \\
&= F(z_3) - F(z_1) \\
&= (1+i-1)^2 - (1-1)^3 \\
&= -1.
\end{aligned}$$

Example. Consider

$$\int_\gamma \frac{i}{z-2-2i}\,dz,$$

where $\gamma(t) = e^{it}$, $0 \leq t \leq \pi$.

Possible antiderivative : $F(z) = i\,\mathrm{Log}(z-2-2i)$. This is analytic on $\mathbb{C}\backslash\{a+2+2i \,:\, a \leq 0\}$.

Let $\Omega := \mathbb{C}\backslash\{a+2+i \,:\, a \leq 0\}$. Then $C_1(0) \subset \Omega$. Let $\gamma(t) = e^{it}$ $(0 \leq t \leq \pi)$ so that $z_1 = \gamma(0) = 1$ and $z_2 = \gamma(\pi) = -1$.

By Theorem 3.3.4,

$$\begin{aligned}\int_{C_1(0)} \frac{i}{z-2-i}\,dz &= i\,\mathrm{Log}(z_2-2-2i) - i\,\mathrm{Log}(z_1-2-2i) \\ &= i\,\mathrm{Log}(-3-2i) - i\,\mathrm{Log}(-1-2i) \\ &\approx 0.5191 - i0.2075\end{aligned}$$

Example. Consider

$$\int_{C_1(0)} \frac{1}{z^n}\, dz$$

where $n \neq 1$.

Possible antiderivative : $F(z) = \frac{-1}{(n-1)z^{n-1}}$. It is analytic on $\Omega = \mathbb{C}\backslash\{0\}$. Notice that this is a region and contains $C_1(0)$.

By Theorem 3.3.4,

$$\int_{C_1(0)} \frac{1}{z^n}\, dz = F(z_2) - F(z_1) = \frac{-1}{(n-1)z_2^{n-1}} + \frac{1}{(n-1)z_1^{n-1}}.$$

However, $C_1(0)$ is a closed curve, so $z_1 = z_2$. Hence

$$\int_{C_1(0)} \frac{1}{z^n} = 0.$$