M444 – Complex Analysis

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Chapter 3

Section 3.8: Cauchy's Integral Formula

Lemma

Let γ be a positively oriented Jordan curve and let $z_0 \notin \gamma$. Then

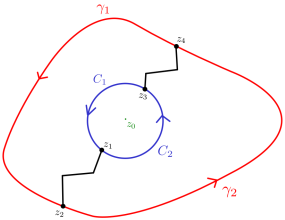
$$\int_{\gamma} \frac{1}{z - z_0}\, dz = \begin{cases} 0 & z_0 \in \Omega^+ \\ 2\pi i & z_0 \in \Omega^- \end{cases}$$

Proof. Assume $z_0 \in \Omega^+$. Then $f(z) = (z - z_0)^{-1}$ is analytic on $\Omega^- \cup \gamma$ and by Cauchy's Theorem

$$\int_{\gamma} \frac{1}{z - z_0}\, dz = 0.$$

Assume $z_0 \in \Omega^-$.

- Consider $B_r(z_0) \subset \Omega^-$ and let $C := C_r(z_0)$ be its boundary.
- Let $[z_1, \ldots, z_2]$ and $[z_3, \ldots, z_4]$ be two different polygonal curves with starting point on C and terminal point on γ.



Let $f(z) = (z - z_0)^{-1}$. By Cauchy's Theorem, we get

$$\int_{\Gamma_1} \frac{1}{z - z_0} \, dz = 0 \quad \text{and} \quad \int_{\Gamma_2} \frac{1}{z - z_0} \, dz = 0.$$

Therefore,

$$\int_{\Gamma_1} f(z) \, dz + \int_{\Gamma_2} f(z) \, dz = 0.$$

But

$$\int_{\Gamma_1} f(z) \, dz = \int_{\gamma_1} f(z) \, dz + \int_{[z_2, \ldots, z_1]} f(z) \, dz - \int_{C_1} f(z) \, dz + \int_{[z_3, \ldots, z_4]} f(z) \, dz$$

and

$$\int_{\Gamma_2} f(z) \, dz = \int_{\gamma_2} f(z) \, dz + \int_{[z_4, \ldots, z_3]} f(z) \, dz - \int_{C_2} f(z) \, dz + \int_{[z_1, \ldots, z_2]} f(z) \, dz$$

Adding up :

$$\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz - \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$

and so

$$\int_{\gamma} f(z)\,dz - \int_{C} f(z)\,dz = 0.$$

This implies

$$\int_{\gamma} \frac{1}{z - z_0}\,dz = \int_{C} \frac{1}{z - z_0}\,dz = i\int_0^{2\pi} dt = 2\pi i.$$

This ends the proof. □

Theorem

Assume that

1. f is an analytic function on a region U.
2. γ is a Jordan curve such that $\Omega^- \cup \gamma \subset U$.

Then, for any $w \in \Omega^-$, we have

$$f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - w}\, dz.$$

Proof. There is a disk $B_R(w) \subset \Omega^-$ and for any $0 < r \leq R$, we have

$$\int_\gamma \frac{f(z)}{z - w}\, dz = \int_{C_r(w)} \frac{f(z)}{z - w}\, dz.$$

<u>Goal :</u> To show the right-hand side is $2\pi i f(w)$ if $r \to 0^+$.

We have

$$\int_{C_r(w)} \frac{f(z)}{z-w}\,dz - 2\pi i f(w) = \int_{C_r(w)} \frac{f(z)-f(w)}{z-w}\,dz$$
$$= \int_0^{2\pi} \Big(\frac{f(w+re^{it})-f(w)}{re^{it}}\Big) ire^{it}\,dt$$
$$= i\int_0^{2\pi} f(w+re^{it})-f(w)\,dt.$$

Therefore,

$$\left|\int_{C_r(w)} \frac{f(z)}{z-w}\,dz - 2\pi i f(w)\right| \le \int_0^{2\pi} |f(w+re^{it})-f(w)|\,dt.$$

Because f is continuous, $f(w+re^{it})-f(w)\to 0$ as $r\to 0^+$. Hence,

$$\lim_{r\to 0^+}\int_{C_r(w)} \frac{f(z)}{z-w}\,dz = 2\pi f(w). \qquad \square$$

Example. Consider the integral

$$\int_{C_2(1)} \frac{z^2+3z-1}{(z+3)(z-2)}\,dz.$$

Notice that -3 is outside of the circle $C_2(1)$ and hence the function

$$f(z) = \frac{z^2+3z-1}{z+3}$$

is analytic in the interior of and on the circle $C_2(1)$.

Using Cauchy's Integral Formula,

$$\int_{C_2(1)} \frac{z^2+3z-1}{(z+3)(z-2)}\,dz = \int_{C_2(1)} \frac{\frac{z^2+3z-1}{z+3}}{z-2}\,dz = 2\pi i f(2) = \frac{18\pi i}{5}.$$

Differentiation under the integral sign.

① γ be a Jordan curve with positively orientation and U be a region.
② $\phi : U \times \gamma \to \mathbb{C}$ be function.
③ For any fixed w, $\phi(w, z)$ is continuous in $z \in \gamma$.
④ For any z, $\phi(w, z)$ is analytic in $w \in U$.
⑤ Complex derivative $\frac{d}{dw}\phi(w, z)$ is continuous in $z \in \gamma$.

Then, the function

$$g(w) = \int_\gamma \phi(w, z)\, dz$$

is analytic in U and its derivative is

$$g'(w) = \int_\gamma \frac{d}{dw}\phi(w, z)\, dz.$$

Theorem

① f is analytic on a region U.

② γ is a Jordan curve such that $\Omega^- \cup \gamma \subset U$.

Then f has derivatives of **any order** n at all points w in the interior of γ, all $f^{(n)}$ are analytic in U, and

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-w)^{n+1}}\, dz.$$

Proof. For $n = 0$, this is the Cauchy's Integral Formula :

$$f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-w}\, dz.$$

Notice that $\phi(w,z) = \frac{1}{2\pi i}\frac{f(z)}{z-w}$ is

- continuous in $z \in \gamma$.
- analytic in $w \in \Omega^-$.

Also

$$\frac{d}{dw}\phi(w,z) = \frac{1}{2\pi i}\frac{f(z)}{(w-z)^2}$$

is continuous on γ.

Therefore, differentiating under the integral (DUI), we get

$$f'(w) = \int_\gamma \frac{d}{dw}\phi(w,z)\,dz = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-w)^2}\,dz.$$

Notice that f' is analytic (apply the first part of DUI).

Reapply differentiating under the integral on the function

- $\phi(w,z) = \frac{1}{2\pi i}\frac{f(z)}{(z-w)^2} \quad \Rightarrow \quad f''(w) = \frac{2}{2\pi i}\int_\gamma \frac{f(z)}{(z-w)^3}\,dz.$
- $\phi(w,z) = \frac{2}{2\pi i}\frac{f(z)}{(z-w)^3} \quad \Rightarrow \quad f^{(3)}(w) = \frac{3!}{2\pi i}\int_\gamma \frac{f(z)}{(z-w)^4}\,dz.$

To complete the proof, use induction. □