M444 – Complex Analysis

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Chapter 5

Section 5.1: Cauchy's Residue Theorem

Let f be analytic in $A_{0,R}(z_0)$. Then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \quad 0 < |z - z_0| < R.$$

If $C_r(z_0)$ be a circle with $0 < r < R$. From the uniform convergence of the Laurent series of f:

$$\int_{C_r(z_0)} f(z)\, dz = \sum_{n=-\infty}^{\infty} a_n \int_{C_r(z_0)} (z - z_0)^n\, dz = a_{-1} \int_{C_r(z_0)} \frac{1}{z - z_0}\, dz.$$

Hence

$$a_{-1} = \frac{1}{2\pi i} \int_{C_r(z_0)} f(z)\, dz.$$

Definition 5.1.1 (Residue)

The coefficient a_{-1} is called the **residue** of f at z_0.

Notation: $a_{-1} = \mathrm{Res}(f, z_0)$ or simply $a_{-1} = \mathrm{Res}(z_0)$.

Theorem 5.1.2 (Cauchy's Residue Theorem)

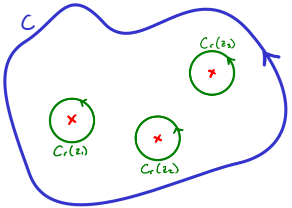
① C is a simple closed positively oriented path.

② f is analytic on the inside and on C, except at finitely many points $z_1, z_2, \ldots, z_n \in \Omega^-$ (the interior of C).

Then

$$\int_C f(z)\, dz = 2\pi i \sum_{j=1}^{n} \text{Res}(f, z_j).$$

Proof.



- Let $C_r(z_j)$ be small circles.
- Using Cauchy's Theorem, we get that

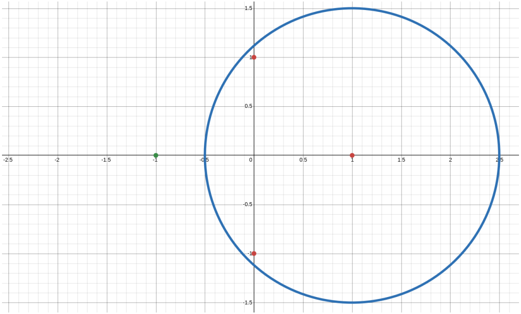
$$\int_C f(z)\, dz = \sum_{j=1}^{n} \int_{C_r(z_j)} f(z)\, dz$$

$$= 2\pi i \sum_{j=1}^{n} \text{Res}(f, z_j). \quad \square$$

Example. Find

$$\int_{C_{3/2}(1)} \frac{1}{z^4 - 1}\, dz.$$

The function $f(z) = \frac{1}{z^4 - 1}$ has singularities at ± 1 and $\pm i$.



Notice that -1 is not in the interior of $C_{3/2}(1)$. Therefore, by Cauchy's Residue Theorem,

$$\int_{C_{3/2}(1)} \frac{1}{z^4 - 1}\, dz = 2\pi i(\mathrm{Res}(f, 1) + \mathrm{Res}(f, i) + \mathrm{Res}(f, -i)).$$

① $z_0 = 1$ is a pole of order $m = 1$. Therefore, we can write

$$f(z) = \frac{a_{-1}}{z-1} + a_0 + a_1(z-1) + \cdots = \frac{a_{-1}}{z-z_0} + h(z)$$

where h is analytic at 1. Therefore

$$(z-1)f(z) = a_{-1} + (z-1)h(z) \quad \Rightarrow \quad \mathrm{Res}(f,1) = \lim_{z\to 1}(z-1)f(z).$$

Replacing the expression of f, we get

$$\mathrm{Res}(f,1) = \lim_{z\to 1}\frac{z-1}{z^4-1} = \lim_{z\to 1}\frac{1}{(z^3+z^2+z+1)} = \frac{1}{4}.$$

Proposition 5.1.3 (i)

If f is an analytic function with a pole of order $m = 1$ at z_0, then

$$\mathrm{Res}(f,z_0) = \lim_{z\to z_0}(z-z_0)f(z).$$

② $z_0 = i$ is also a pole of order $m = 1$. So,

$$\mathrm{Res}(f, i) = \lim_{z \to i} \frac{z - i}{z^4 - 1} = \lim_{z \to i} \frac{1}{4z^3} = \frac{i}{4}.$$

③ $z_0 = -i$ is also a pole of order $m = 1$. So,

$$\mathrm{Res}(f, -i) = \lim_{z \to -i} \frac{z + i}{z^4 - 1} = \lim_{z \to -i} \frac{1}{4z^3} = -\frac{i}{4}.$$

④ Collecting everything together, we get

$$\int_{C_{3/2}(1)} \frac{1}{z^4 - 1}\, dz = 2\pi i \Big(\frac{1}{4} + \frac{i}{4} - \frac{i}{4} \Big) = \frac{\pi i}{2}.$$

Theorem 5.1.6 (Pole of Higher Order)

Assume that z_0 is a pole of order $m \geq 1$ of f. Then,

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \Big((z - z_0)^m f(z) \Big).$$

Example. Find residue at the pole of $f(z) = \left(\frac{z-1}{z+3i}\right)^2$.

Notice that $z = -3i$ is a pole of order $m = 2$. Therefore,

$$\begin{aligned}
\operatorname{Res}(f, -3i) &= \lim_{z \to -3i} \frac{d^{2-1}}{dz^{2-1}} \Big((z+3i)^2 \frac{(z-1)^2}{(z+3i)^2} \Big) \\
&= \lim_{z \to -3i} \frac{d}{dz} (z-1)^2 = \lim_{z \to -3i} 2(z-1) \\
&= 2(-3i - 1) = -6i - 2.
\end{aligned}$$