M444 – Complex Analysis

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Chapter 5

Section 5.2: Definite Integrals of Trigonometric Functions

Example. Compute the definite integral

$$\int_0^{2\pi} \frac{1}{2 - \cos\theta}\, d\theta.$$

<u>Trick:</u> Change of variable. Set $z = e^{i\theta}$, where $0 \leq \theta \leq 2\pi$. Then

$$\frac{dz}{d\theta} = ie^{i\theta} \quad \Rightarrow \quad dz = ie^{i\theta}\, d\theta \quad \Rightarrow \quad -\frac{i}{z}\, dz = d\theta.$$

Now, we get

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{e^{2i\theta} + 1}{2e^{i\theta}} = \frac{z^2 + 1}{2z}.$$

Hence,

$$\frac{1}{2 - \cos\theta} = \frac{1}{2 - \frac{z^2+1}{2z}} = \frac{2z}{4z - z^2 - 1}.$$

Replacing that in the integral:

$$\int_0^{2\pi} \frac{1}{2-\cos\theta}\,d\theta = \int_{C_1(0)} \Big(\frac{2z}{4z-z^2-1}\Big)\Big(-\frac{i}{z}\Big)\,dz.$$
$$= i\int_{C_1(0)} \frac{2}{z^2-4z+1}\,dz.$$

The singularity of $f(z) = \frac{2}{z^2-4z+1}$ are $z_1 = 2+\sqrt{3}$ and $z_2 = 2-\sqrt{3}$.

Only z_2 is inside $C_1(0)$! Hence, from Cauchy's Residue Theorem

$$\int_0^{2\pi} \frac{1}{2-\cos\theta}\,d\theta = i(2\pi i\,\mathrm{Res}(f,2-\sqrt{3})) = -2\pi\,\mathrm{Res}(f,2-\sqrt{3}).$$

We find that $\mathrm{Res}(f,2-\sqrt{3}) = -1/\sqrt{3}$. Therefore

$$\int_0^{2\pi} \frac{1}{2-\cos\theta}\,d\theta = \frac{2\pi}{\sqrt{3}}.$$

Goal. Evaluate

$$\int_0^{2\pi} F(\sin\theta, \cos\theta)\, d\theta.$$

Trick: We set

$$z = e^{i\theta} \quad \Rightarrow \quad -\frac{i}{z}\, dz = d\theta$$

and use the fact that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z^2+1}{2z}$$

and

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z^2-1}{2iz}$$

Then we substitute $\cos\theta$, $\sin\theta$, and $d\theta$ to rewrite the integral as a complex path integral:

$$-i \int_{C_1(0)} F\Big(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\Big)\, \frac{dz}{z}.$$

Practice Problems. 2, 4, 5.