M444 – Complex Analysis

Pierre-Olivier Parisé

University of Hawai'i at Mānoa
Chapter 5

Section 5.6: Summing Series by Residues

Lemma

Let f be analytic at some integer k. Then

$$\operatorname{Res}(f(z)\cot(\pi z), k) = \frac{f(k)}{\pi}.$$

Proof. Notice that $\cot(\pi z)$ has singularities at every integers, in particular at $z = k$. It is a simple pole because:

$$\lim_{z\to k}(z-k)\cot(\pi z) = \lim_{z\to k}\frac{(z-k)\cos(\pi z)}{\sin(\pi z)} = \frac{\cos(\pi k)}{\frac{d}{dz}(\sin(\pi z))\big|_{z=k}} = \frac{1}{\pi}.$$

Hence

$$\begin{aligned}\operatorname{Res}(f(z)\cot(\pi z), k) &= \lim_{z\to k}(z-k)\frac{f(z)\cos(\pi z)}{\sin(\pi z)} \\ &= \lim_{z\to k} f(z)\lim_{z\to k}\frac{(z-k)\cos(\pi z)}{\sin(\pi z)} = \frac{f(k)}{\pi}.\end{aligned}$$

$\square$

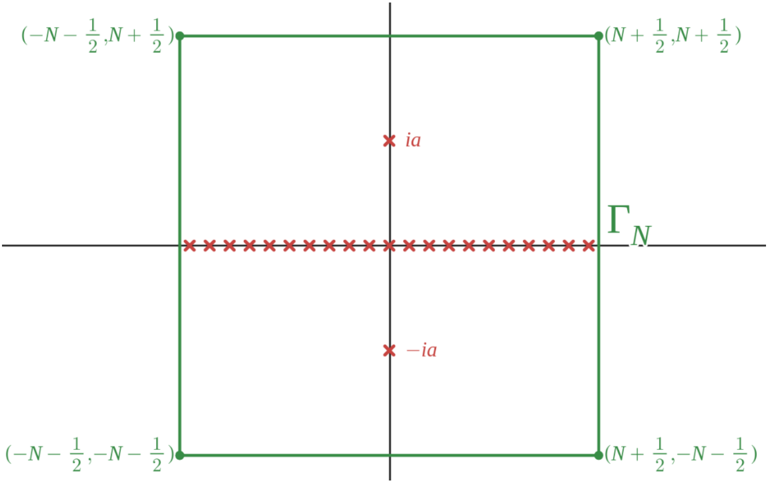
Example. For any $a \in \mathbb{R}\backslash\{0\}$, evaluate

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2 + a^2} = \lim_{N\to\infty} \sum_{k=-N}^{N} \frac{1}{k^2 + a^2}.$$

First, notice that $f(z) = \frac{1}{z^2+a^2}$

- is analytic at every integer k.
- has simple poles at $w_1 = -ia$ and $w_2 = ia$.
- $f(k) = \frac{1}{k^2+a^2} = \pi \operatorname{Res}(f(z)\cot(\pi z), k)$.

Consider the following path Γ_N, for $N \geq 1$ an integer.



By Cauchy's Residue Theorem:

$$\int_{\Gamma_N} f(z)\cot(\pi z)\,dz = 2\pi i\Big(\operatorname{Res}(f(z)\cot(\pi z), -ia) + \operatorname{Res}(f(z)\cot(\pi z), ia)\Big)$$
$$+ 2\pi i\Big(\sum_{k=-N}^{N} \operatorname{Res}(f(z)\cot(\pi z), k)\Big).$$

We have

1. $\operatorname{Res}(f(z)\cot(\pi z), -ia) = -\frac{\cot(-i\pi a)}{2ia} = \frac{\cot(i\pi a)}{2ia} = \frac{i\coth(\pi a)}{2ia} = \frac{\coth(\pi a)}{2a}$.
2. $\operatorname{Res}(f(z)\cot(\pi z), ia) = \frac{\cot(i\pi a)}{2ia} = \frac{\coth(\pi a)}{2a}$.
3. $\operatorname{Res}(f(z)\cot(\pi z), k) = \frac{1}{\pi}\frac{1}{k^2+a^2}$.

Hence

$$\int_{\Gamma_N} f(z)\cot(\pi z)\,dz = \frac{2\pi i\coth(\pi a)}{a} + 2i\sum_{k=-N}^{N}\frac{1}{k^2+a^2}.$$

It would be nice if

$$\lim_{N\to\infty} \int_{\Gamma_N} f(z)\cot(\pi z)\,dz = 0.$$

In fact, we can show that

$$z \in \Gamma_N \quad \Rightarrow \quad |\cot(\pi z)| \leq 2.$$

Also, for $z \in \Gamma_N$

$$|f(z)| = \frac{1}{|z^2 + a^2|} \leq \frac{1}{|z|^2 - |a|^2} \leq \frac{1}{(N - \frac{1}{2})^2 - |a|^2}.$$

Therefore

$$\Big| \int_{\Gamma_N} f(z)\cot(\pi z)\,dz \Big| \leq \ell(\Gamma_N) \max_{\Gamma_N} |f(z)\cot(\pi z)| \leq \frac{4(2N-1)(2)}{(N-1/2)^2 - |a|^2}.$$

Hence, from the last slide:

$$\lim_{N\to\infty} \left| \int_{\Gamma_N} f(z)\cot(\pi z)\,dz \right| \le \lim_{N\to\infty} \frac{8(2N-1)}{(N-1/2)^2 - |a|^2} = 0$$

and

$$\lim_{N\to\infty} \int_{\Gamma_N} f(z)\cot(\pi z)\,dz = 0.$$

Now, we obtain

$$\lim_{N\to\infty} \int_{\Gamma_N} f(z)\cot(\pi z)\,dz = \frac{2\pi i\coth(\pi a)}{a} + 2i \lim_{N\to\infty} \sum_{k=-N}^{N} \frac{1}{k^2+a^2}$$

$$\iff \qquad 0 = \frac{\pi\coth(\pi a)}{a} + \sum_{k=-\infty}^{\infty} \frac{1}{k^2+a^2}$$

Hence

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2+a^2} = -\frac{\pi\coth(\pi a)}{a}.$$

Proposition 5.6.2

Suppose that

1. $f(z) = \frac{p(z)}{q(z)}$ is a rational function with $\deg q \geq 2 + \deg p$.
2. f has no poles at the integers.
3. f has poles at z_1, z_2, ..., z_n.

Then

$$\sum_{k=-\infty}^{\infty} f(k) = -\pi \sum_{j=1}^{n} \operatorname{Res}(f(z) \cot(\pi z), z_j).$$

How do we obtain the value of $\sum_{k=1}^{\infty} \frac{1}{k^2}$?

We start from

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2 + a^2} = \sum_{k=-1}^{\infty} \frac{1}{k^2 + a^2} + \frac{1}{a^2} + \sum_{k=1}^{\infty} \frac{1}{k^2 + a^2}$$

which implies that

$$\frac{\pi \coth(\pi a)}{a} = \sum_{k=-\infty}^{\infty} \frac{1}{k^2 + a^2} = \frac{1}{a^2} + 2\sum_{k=1}^{\infty} \frac{1}{k^2 + a^2}.$$

Hence

$$\sum_{k=1}^{\infty} \frac{1}{k^2 + a^2} = \frac{a\pi \coth(\pi a) - 1}{2a^2}.$$

Let $-1 < a < 1$. Then

$$\sum_{k=1}^{\infty} \frac{1}{k^2 + a^2} \leq \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

By the Weierstrass M-test, $g(a) = \sum_{k=1}^{\infty} \frac{1}{k^2+a^2}$ converges uniformly on $(-a, a)$ and therefore

$$\lim_{a \to 0} \sum_{k=1}^{\infty} \frac{1}{k^2 + a^2} = \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Therefore,

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \lim_{a \to 0} \frac{a\pi \coth(\pi a) - 1}{2a^2} = \frac{\pi^2}{6}.$$

Question: Can you evaluate $\sum_{k=1}^{\infty} \frac{1}{k^3}$?

Proposition (see Exercise 13)

Assume that

1. $f = \frac{p}{q}$ is a rational function with $\deg p + 2 \leq \deg q$.
2. f has no pole at the non-zero integers.
3. f has poles at z_1, z_2, ..., z_n (might be at 0).

Then

$$\sum_{k=1}^{\infty} f(-k) + \sum_{k=1}^{\infty} f(k) = -\pi \sum_{j=1}^{n} \operatorname{Res}(f(z)\cot(\pi z), z_j).$$

Proposition (see Exercise 18)

Assume that

1. $f = \frac{p}{q}$ is a rational function with $\deg p + 2 \leq \deg q$.
2. f has no pole at the non-zero integers.
3. f has poles at z_1, z_2, ..., z_n (might be at 0).

Then

$$\sum_{k=1}^{\infty}(-1)^k f(-k) + \sum_{k=1}^{\infty}(-1)^k f(k) = -\pi \sum_{j=1}^{\infty} \operatorname{Res}(f(z)\csc(\pi z), z_j).$$