Section 1.1 M444

Problems Solution Pierre Parisé

Problem 3

$$\frac{i(3+i)}{3} = \frac{3i+i^2}{3} = \frac{3i-1}{3} = \boxed{\frac{-1}{3} + 3i}.$$

Problem 5

$$\left(\overline{2-i}\right)^2 = (2+i)^2 = (2+i)(2+i) = 4-1+(2+2)i$$
$$= \boxed{3+4i}$$

Problem 10

$$\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^3 = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$= \left[\left(\frac{1}{4} - \frac{3}{4}\right) + \frac{2\sqrt{3}}{4}i\right]\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$= \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$= \left(-\frac{1}{4} - \frac{3}{4}\right) + \left(\frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{4}i\right)$$

$$= \boxed{-1}.$$

Problem 11

$$(2i)^5 = 2^5 i^5 = 32\, i^2 i^2 i = 32(-1)(-1)i$$
$$= \boxed{32i}$$

Problem 12

$$i^{12} + i^{25} - 7i^{111} = (i^2)^6 + (i^2)^{12} i - 7(i^2)^{55} i$$
$$= (-1)^6 + (-1)^{12} i - 7(-1)^{55} i$$
$$= 1 + i + 7i$$
$$= \boxed{1 + 8i}$$

Problem 18

$$\frac{101+i}{100+i} = \frac{101+i}{100+i}\,\frac{\overline{100+i}}{\overline{100+i}} = \frac{(101+i)(100-i)}{(100+i)(100-i)}$$
$$= \frac{10100 + 1 - i}{10000 + 1}$$
$$= \frac{10101 - i}{10001} = \boxed{\frac{10101}{10001} - \frac{i}{10001}}$$

Problem 22

(a) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

$$Re(z_1 + z_2) = Re(x_1 + x_2 + i(y_1 + y_2))$$
$$= x_1 + x_2 = Re(z_1) + Re(z_2).$$

$$Re(z_1 - z_2) = Re(x_1 - x_2 + i(y_1 - y_2))$$
$$= x_1 - x_2 = Re(z_1) - Re(z_2).$$

(b) Let $z_1 = i$ and $z_2 = i$. Then

$z_1 z_2 = -1$ so $Re(z_1 z_2) = -1$.

But, $Re(z_1) Re(z_2) = 0 \cdot 0 = 0$

$\Rightarrow Re(z_1 z_2) \neq Re(z_1) Re(z_2)$.

(c) ($\Rightarrow$) $Re(z_1 z_2) = x_1 x_2 - y_1 y_2$

and $Re(z_1) = x_1$ and $Re(z_2) = x_2$. Assume that $Re(z_1 z_2) = Re(z_1) Re(z_2)$. This implies that

$$x_1 x_2 - y_1 y_2 = x_1 x_2 \Rightarrow y_1 y_2 = 0.$$

Thus, $y_1 = 0$ or $y_2 = 0$. Thus, $z_1 = x_1$ or $z_2 = x_2$. Hence, z_1 is a real number or z_2 is a real number.

($\Leftarrow$) Assume that $z_1 = x_1$, a real number.

Then $\operatorname{Re}(z_1 z_2) = x_1 x_2 - y_1 y_2 = x_1 x_2$

$$= \operatorname{Re}(z_1)\operatorname{Re}(z_2).$$

Assume $z_2 = x_2$, a real number. Then

$$\operatorname{Re}(z_1 z_2) = x_1 x_2 - y_1 y_2 = x_1 x_2$$
$$= \operatorname{Re}(z_1)\operatorname{Re}(z_2).$$

Therefore, in each case,

$$\operatorname{Re}(z_1 z_2) = \operatorname{Re}(z_1)\operatorname{Re}(z_2).$$ $\square$