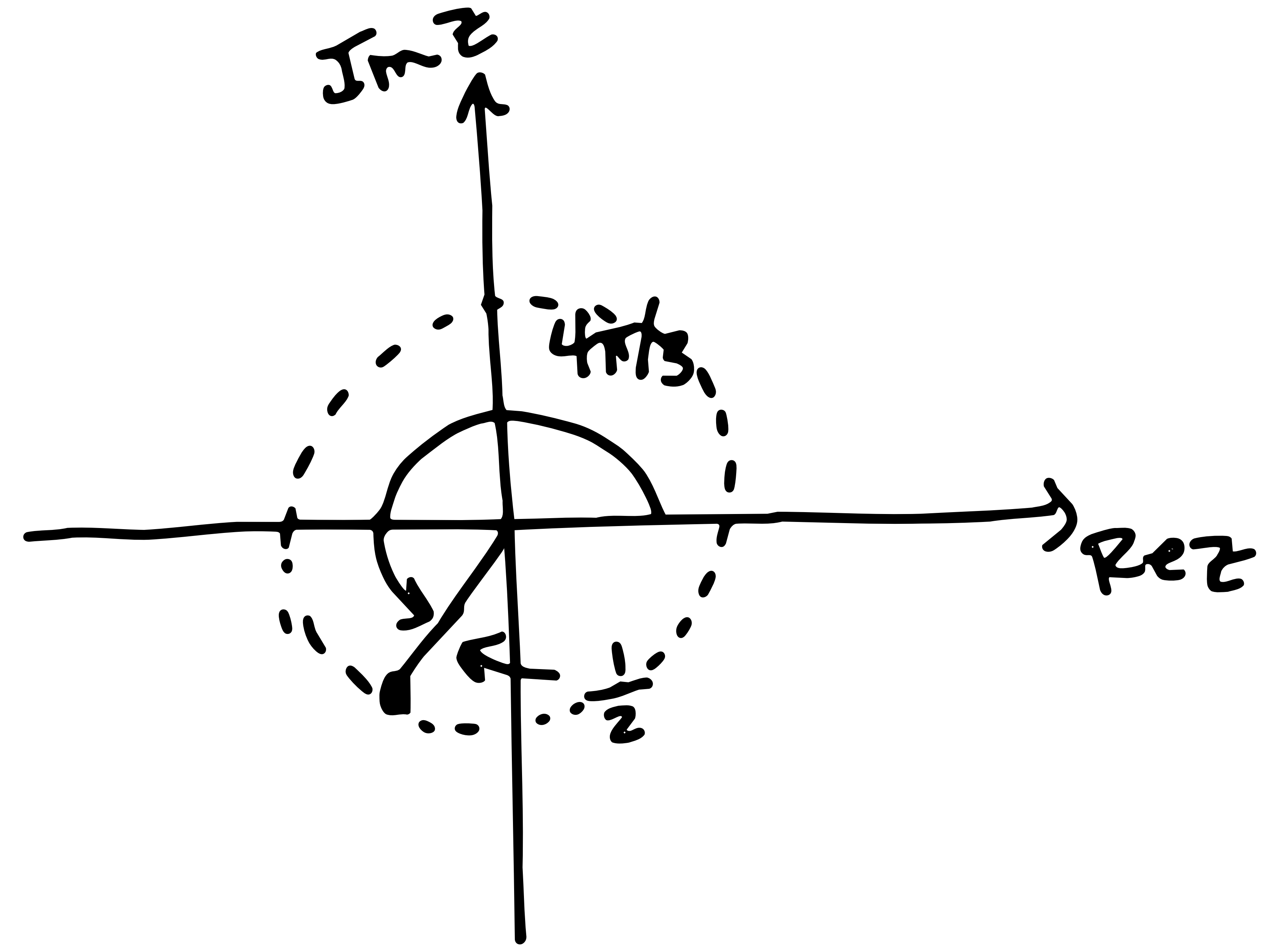
Section 1.3 M444

Problems Solution

Problem 3

$$\frac{64\pi}{3} = \frac{3\cdot 20\pi + 4\pi}{3} = 20\pi + \underbrace{\frac{4\pi}{3}}_{\theta\,:\ \text{argument}}.$$

radius $= \frac{1}{2}$.



Problem 6

$$r = \left(-\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{4} = 1.$$

$$\cos\theta = -\frac{\sqrt{3}}{2} \quad \text{and} \quad \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{5\pi}{6}$$

Thus, $z = 1\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$.

Problem 10

Write $\dfrac{1+i}{1+\sqrt{3}\,i} = \dfrac{z_1}{z_2}$.

Then, $r_1 = |z_1| = \sqrt{2}$ and $\theta = \dfrac{\pi}{4}$

$r_2 = |z_2| = 2$ and $\theta = \dfrac{\pi}{3}$.

thus,

$$\frac{1+i}{1+\sqrt{3}\,i} = \frac{\sqrt{2}\left(\cos(\pi/4) + i\sin(\pi/4)\right)}{2\left(\cos(\pi/3) + i\sin(\pi/3)\right)}$$

$$= \frac{\sqrt{2}}{2}\left(\cos\left(\frac{\pi}{4} - \frac{\pi}{3}\right) + i\sin\left(\pi/4 - \pi/3\right)\right)$$

$$= \boxed{\frac{\sqrt{2}}{2}\left(\cos\left(\frac{-\pi}{12}\right) + i\sin\left(\frac{-\pi}{12}\right)\right)}.$$

Problem 11

We have

$$\frac{1+i}{1-i} = \frac{\sqrt{2}\left(\cos \pi/4 + i\sin(\pi/4)\right)}{\sqrt{2}\left(\cos(-\pi/4) + i\sin(-\pi/4)\right)}$$

$$= \frac{\sqrt{2}}{\sqrt{2}}\left(\cos\left(\frac{\pi}{4} + \pi/4\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi}{4}\right)\right)$$

$$= \boxed{\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)}$$

Problem 14

$$r = \sqrt{9 + 32^2} = \sqrt{9 + 1024} = \sqrt{1035} \approx 32.17.$$

$$\cos\theta = \frac{-3}{\sqrt{1035}} \quad \text{and} \quad \sin\theta = \frac{-32}{\sqrt{1035}}$$

$$\Rightarrow \quad \tan\theta = \frac{-3}{-32} = \frac{3}{32}$$

$$\Rightarrow \quad \theta = 0.09 - \pi \longrightarrow \text{because } \begin{array}{l} \operatorname{Re} z < 0 \\ \operatorname{Im} z < 0. \end{array}$$

$$\Rightarrow \quad \theta = -3.048.$$

Since $\theta \in [-\pi, \pi) \Rightarrow \boxed{\operatorname{Arg} z = \theta = -3.048}$

In general, we have

$$\arg z = \{-3.048 + 2k\pi : k \in \mathbb{Z}\}.$$

Problem 19

Write $1 - i = \sqrt{2}\left(\cos(-\pi/4) + i\sin(-\pi/4)\right)$

and $1 + i = \sqrt{2}\left(\cos \pi/4 + i \sin \pi/4\right)$

So, $$\frac{1-i}{1+i} = \cos(-\pi/4 - \pi/4) + i \sin(-\pi/4 - \pi/4)$$

$$= \cos(-\pi/2) + i \sin(-\pi/2) = -i$$

$$\Rightarrow \left(\frac{1-i}{1+i}\right)^{10} = (-i)^{10} = (-1)^{10}\,(i^2)^5 = -1$$

$$\Rightarrow \boxed{\left(\frac{1-i}{1+i}\right)^{10} = \cos(-\pi) + i\sin(-\pi).}$$

Problem 21

$$1+i = \sqrt{2}\left(\cos(\pi/4) + i\sin(\pi/4)\right)$$

$$\Rightarrow (1+i)^{30} = (\sqrt{2})^{30}\left(\cos\left(\frac{30\pi}{4}\right) + i\sin\left(\frac{30\pi}{4}\right)\right)$$

$$= \left(\left(\sqrt{2}\right)^2\right)^{15}\left(\cos\left(8\pi - \frac{\pi}{2}\right) + i\sin\left(8\pi - \frac{\pi}{2}\right)\right)$$

$$= 2^{15}\left[\cos\left(\frac{-\pi}{2}\right) + i\sin\left(\frac{-\pi}{2}\right)\right]$$

$$= 2^{15}\left[\,0 - i\,\right]$$

$$\Rightarrow \boxed{(1+i)^{30} = -32768\,i}$$

Problem 25

(a) Take $z_1 = i$ and $z_2 = i$. Then

$$z_1 z_2 = i^2 = -1 \longrightarrow \operatorname{Arg}(z_1 z_2) = -\pi.$$

But, $\operatorname{Arg} z_1 = \frac{\pi}{2} = \operatorname{Arg}(z_2)$

$$\Rightarrow \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) = \pi \neq \operatorname{Arg}(z_1 z_2).$$

(b) Take $z_1 = 1$ and $z_2 = -1$. Then

$$\operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \operatorname{Arg}(-1) = -\pi.$$

But $\operatorname{Arg}(z_1) = \pi$ and $\operatorname{Arg} z_2 = -\pi$

$$\Rightarrow \operatorname{Arg}(z_1) - \operatorname{Arg}(z_2) = 2\pi \neq \operatorname{Arg}\left(\frac{z_1}{z_2}\right)$$

(c) Take $z_1 = -1$. Then, $\overline{z_1} = -1 = z_1$

$$\Rightarrow \operatorname{Arg}(\overline{z_1}) = -\pi \neq -\operatorname{Arg}(z_1).$$

(d) Take $z_1 = 1$. Then $\operatorname{Arg} z_1 = 0$ and $-z_1 = -1$

$$\rightarrow \operatorname{Arg}(-z_1) = -\pi \neq \operatorname{Arg} z_1 + \pi.$$

$\square$

Problem 40

Write $-1+i = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

and $z = r(\cos\theta + i\sin\theta)$.

then $z^3 = -1+i$

$$\Longleftrightarrow \quad z_k = \sqrt[3]{\sqrt{2}}\left(\cos\left(\frac{3\pi/4 + 2k\pi}{3}\right) + i\sin\left(\frac{3\pi/4 + 2k\pi}{3}\right)\right).$$

for $k = 0, 1, 2$.

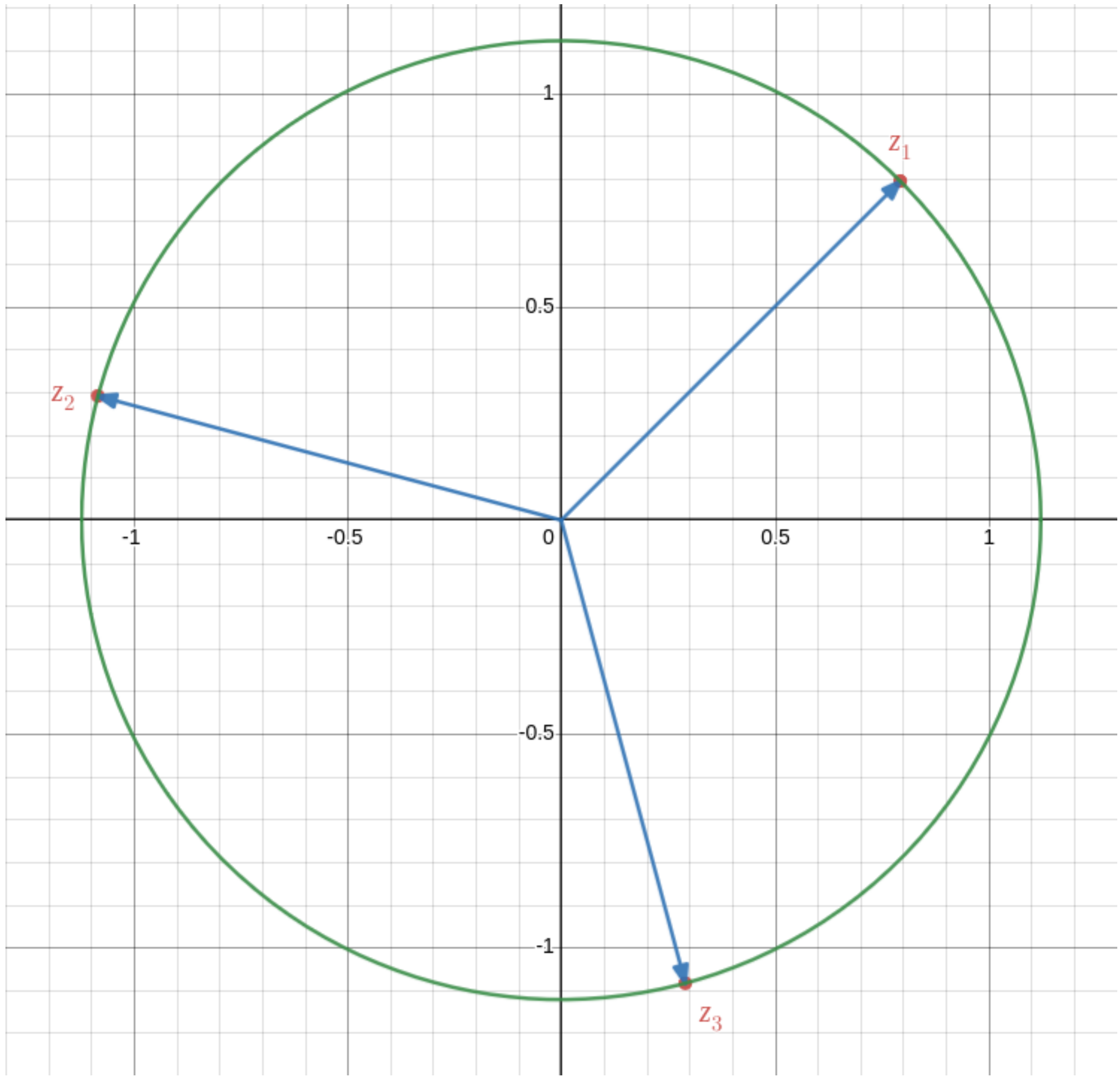
So, the solutions are

$$\boxed{\begin{aligned} z_1 &= \sqrt[6]{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) \\ z_2 &= \sqrt[6]{2}\left(\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right) \\ z_3 &= \sqrt[6]{2}\left(\cos\left(\frac{19\pi}{12}\right) + i\sin\left(\frac{19\pi}{12}\right)\right)\end{aligned}}.$$

(Here, $\frac{\pi}{4} = \frac{3\pi}{12}$)

Here the principal root is z_1, because the argument is

$$\theta = \frac{\operatorname{Arg}(-1+i)}{3}.$$



1
0.5
-0.5
-1
-1
-0.5
0
0.5
1
z_1
z_2
z_3

Problem 45

Let $w = z+2$. Then the equation becomes:

$$w^3 = 3i$$

Let $w = r(\cos\theta + i\sin\theta)$ and

$3i = 3(\cos \pi/2 + i\sin \pi/2)$ where

$\pi/2 = \text{Arg}(3i)$.

Thus,
$$\begin{cases} w_1 = \sqrt[3]{3}\left(\cos \pi/6 + i\sin \pi/6\right) \\ w_2 = \sqrt[3]{3}\left(\cos\left(5\pi/6\right) + i\sin\left(\frac{5\pi}{6}\right)\right) \\ w_3 = \sqrt[3]{3}\left(\cos\left(9\pi/6\right) + i\sin\left(\frac{9\pi}{6}\right)\right) \end{cases}$$

$\frac{3\pi}{2}$

$$\Rightarrow \begin{cases} w_1 = \sqrt[3]{3}\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) \\ w_2 = \sqrt[3]{3}\left(-\frac{\sqrt{3}}{2} + \frac{i}{2}\right) \\ w_3 = \sqrt[3]{3}(-i) = -\sqrt[3]{3}\, i \end{cases}$$

Since $w_k = z_k + 2$, $k = 1, 2, 3$

$$\Rightarrow \boxed{\begin{aligned} z_1 &= \left(\frac{3^{5/6}}{2} - 2\right) + \frac{3^{1/3}}{2} i \\ z_2 &= \left(-\frac{3^{5/6}}{2} - 2\right) + \frac{3^{1/3}}{2} i \\ z_3 &= \left(-2 - 3^{1/3} i\right) \end{aligned}}$$

Problem 57

We have

$$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta \, i\sin\theta + 3\cos\theta \, i^2 \sin^2\theta + i^3 \sin^3\theta$$

$$= \cos^3\theta - 3\cos\theta\sin^2\theta + i\left(3\cos^2\theta\sin\theta - \sin^3\theta\right)$$

By de Moivre's identity:

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$$

$$\Rightarrow \quad \operatorname{Re}(\cos\theta + i\sin\theta)^3 = \operatorname{Re}(\cos 3\theta + i\sin 3\theta)$$

$$\Rightarrow \quad \boxed{\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta} \quad . \ \square$$

Problem 58

From Problem $\boxed{57}$,

$$\operatorname{Im}(\cos\theta + i\sin\theta)^3 = 3\cos^2\theta\sin\theta - \sin^3\theta.$$

From de Moivre's identity

$$\Rightarrow \boxed{\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta}.$$

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