Section 1.4 M444

Problems Solution Pierre Parisé

Problem 3

$z = -1 - i$

$$f(-1-i) = (-1-i)^2 + 2i(-1-i) - 1$$
$$= (-1)^2(1+i)^2 + 2i + 2 - 1$$
$$= 2i + 2i + 2 - 1$$
$$= \boxed{1 + 4i}$$

$z = 1 + i$

$$f(1+i) = (1+i)^2 + 2i(1+i) - 1$$
$$= 2i + 2i - 2 - 1$$
$$= \boxed{-3 + 4i}$$

$z = 0$

$$f(0) = 0^2 + 2i(0) - 1$$
$$= \boxed{-1}$$

$z = 2i$

$$f(z) = (2i)^2 + (2i)(2i) - 1$$
$$= -4 - 4 - 1$$
$$= \boxed{-9}$$

Problem 15

We have

$$f(z) = \frac{z-1}{z+1} = \frac{(z-1)(\overline{z+1})}{(z+1)(\overline{z+1})}$$

$$= \frac{(z-1)(\bar{z}+1)}{(z+1)(\bar{z}+1)}$$

$$= \frac{|z|^2 + z - \bar{z} - 1}{|z|^2 + z + \bar{z} + 1}$$

$$= \frac{|z|^2 - 1 + 2i\,\mathrm{Im}(z)}{|z|^2 + 2\,\mathrm{Re}\, z + 1}$$

This is the imaginary part

This is the real part

$$= \frac{|z|^2-1}{|z|^2 + 2\,\mathrm{Re}\, z + 1} + i\,\frac{2\,\mathrm{Im}\, z}{|z|^2 + 2\,\mathrm{Re}\, z + 1}$$

Thus,

$$\boxed{u(z) = \frac{|z|^2-1}{|z|^2 + 2\,\mathrm{Re}\, z + 1}}$$

and

$$\boxed{v(z) = \frac{2\,\mathrm{Im}\, z}{|z|^2 + 2\,\mathrm{Re}\, z + 1}}$$

Note:

$|z|^2 - 1 \in \mathbb{R}$

$2\,\mathrm{Im}\, z \in \mathbb{R}$

$|z|^2 + 2\,\mathrm{Re}\, z + 1 \in \mathbb{R}$

Problem 19

(a) We don't want $2-i-z=0$. Thus, $z \neq 2-i$. The expression is defined for any $z \in \mathbb{C}$ for which $z \neq 2-i$.

b) No problem. Defined for any $z \in \mathbb{C}$.

Problem 32

Write $z = r(\cos\theta + i\sin\theta)$, $z \neq 0$.

Since $\forall z \in S, r = |z| \geq 1$, we have $z \neq 0$.

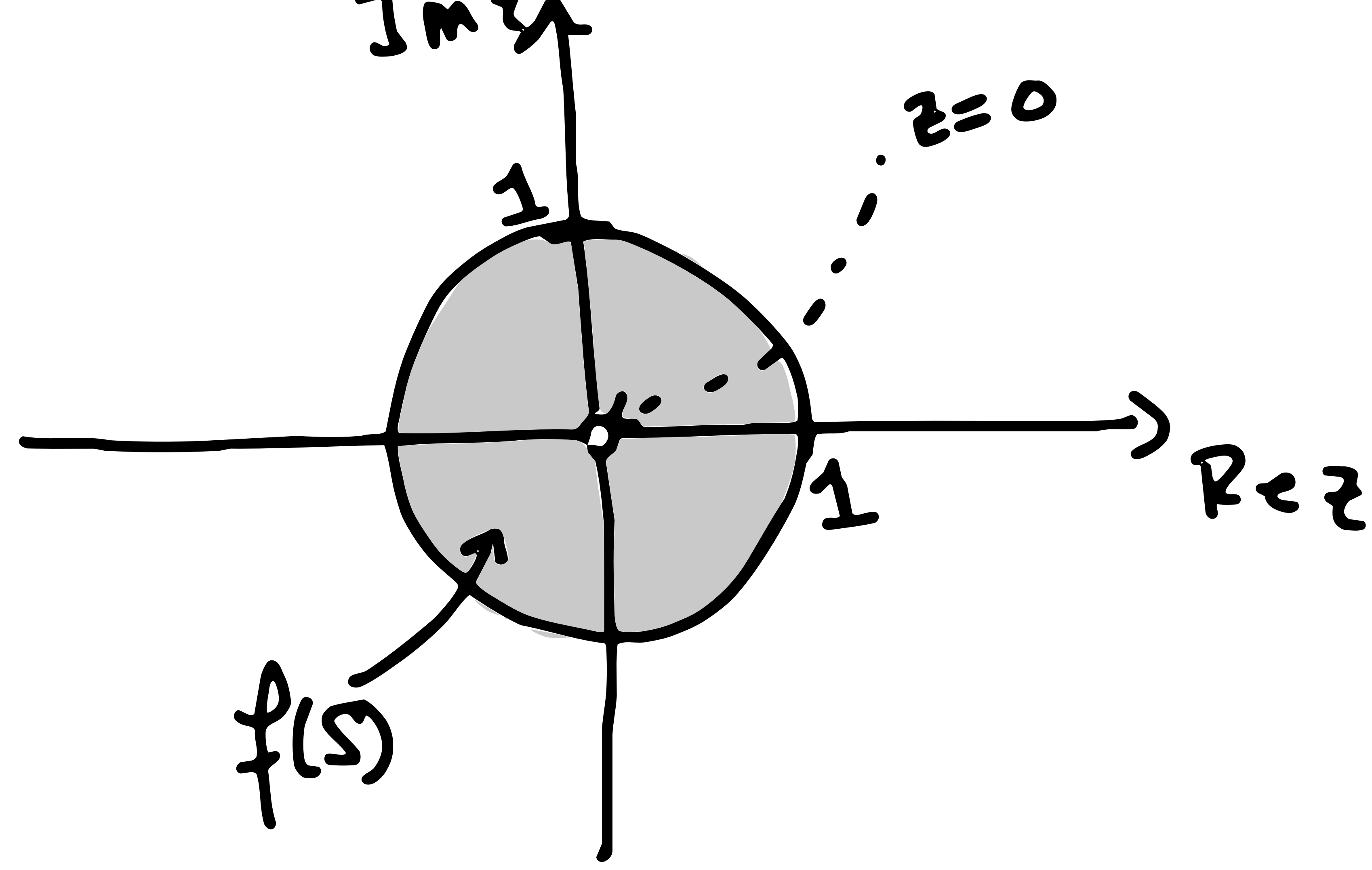
Thus,
$$\frac{1}{z} = \frac{1}{r}\left(\cos(-\theta) + i\sin(-\theta)\right),$$

and $r \geq 1$.

Therefore, $\left|\frac{1}{z}\right| = \frac{1}{r} \leq 1$. Since $|z| \neq \infty$, then $\frac{1}{r} \neq 0$ and thus

$$f(S) = \{\omega \in \mathbb{C} : 0 < |\omega| \leq 1\}.$$

Picture:



Problem 39

Write $S = \{ x+iy : -3 \le x \le 3,\ 0 \le y \le 1 \}$.

and $f(z) = z^2 = (x^2 - y^2) + i(2xy)$.

We will find the image of the boundary.

① Fix $x = x_0 \in [-3, 3]$.

In this case,

$$u = x_0^2 - y^2 \quad \& \quad v = 2x_0 y$$

for $y \in [0,1]$. Thus, when $x_0 \ne 0$, we obtain

$$y = \frac{v}{2x_0}$$

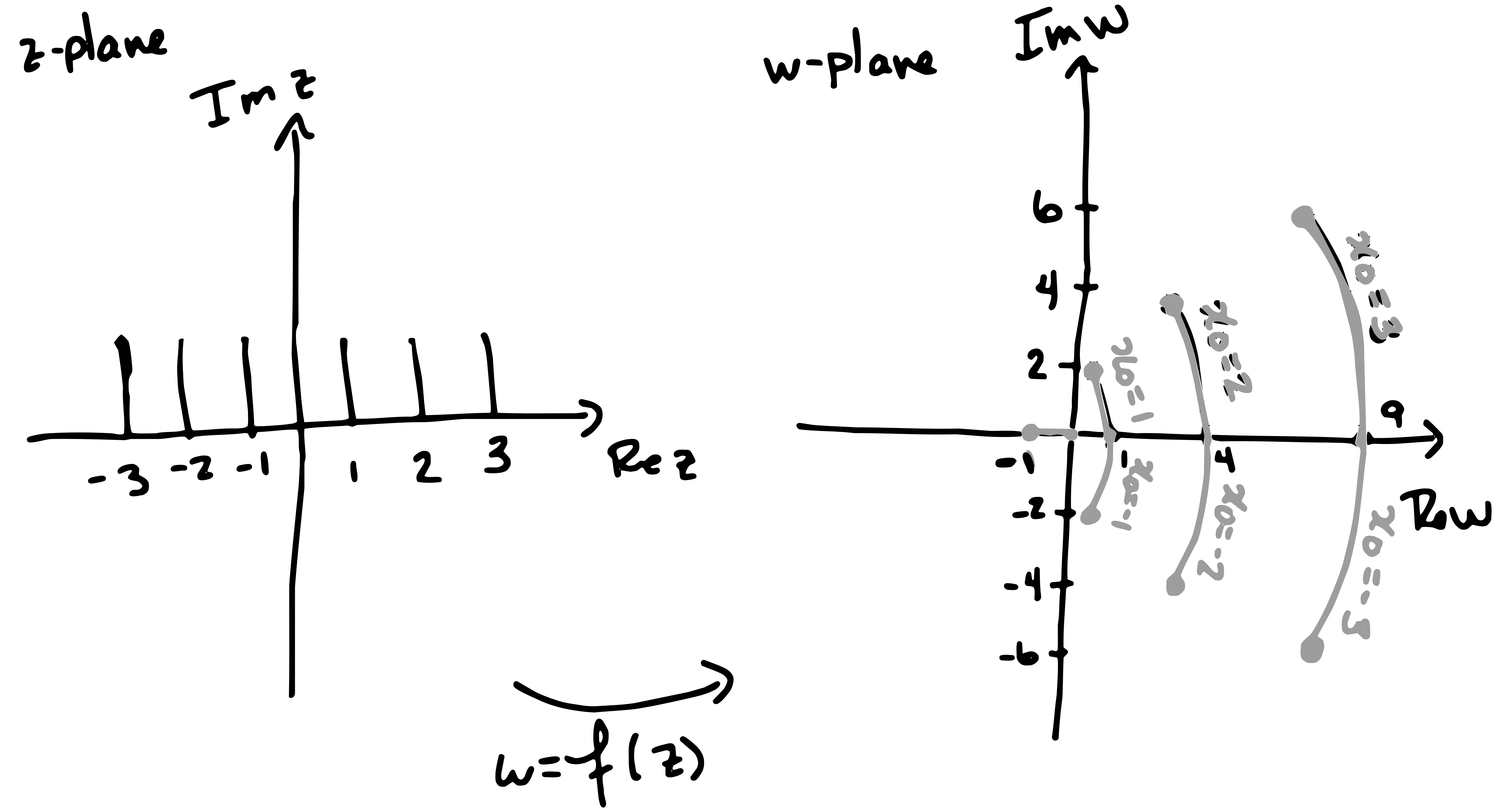
$$\Rightarrow \quad u = x_0^2 - \frac{v^2}{4x_0^2}, \quad 0 \le v \le 2x_0.$$

So each vertical segment $x = x_0$ is mapped to a parabola.

When $x_0 = 0$, then

$$u = -y^2 \quad , \quad v = 0 \quad , \quad y \in [0,1].$$

This is a horizontal line from $(-1,0)$ to $(0,0)$. So the line $x=0$ is mapped to the segment connecting $(-1,0)$ to $(0,0)$.



② Fix $y = y_0 \in [0,1]$

In this case, we have

$$u = x^2 - y_0^2 \quad \text{and} \quad v = 2x y_0$$

for $-3 \le x \le 3$. Isolating x from 2^{nd} eq.

$$\Rightarrow \quad u = \frac{v^2}{4y_0^2} - y_0^2 \quad , \; -6y_0 \le v \le 6y_0 \quad , \; y_0 \neq 0$$

and

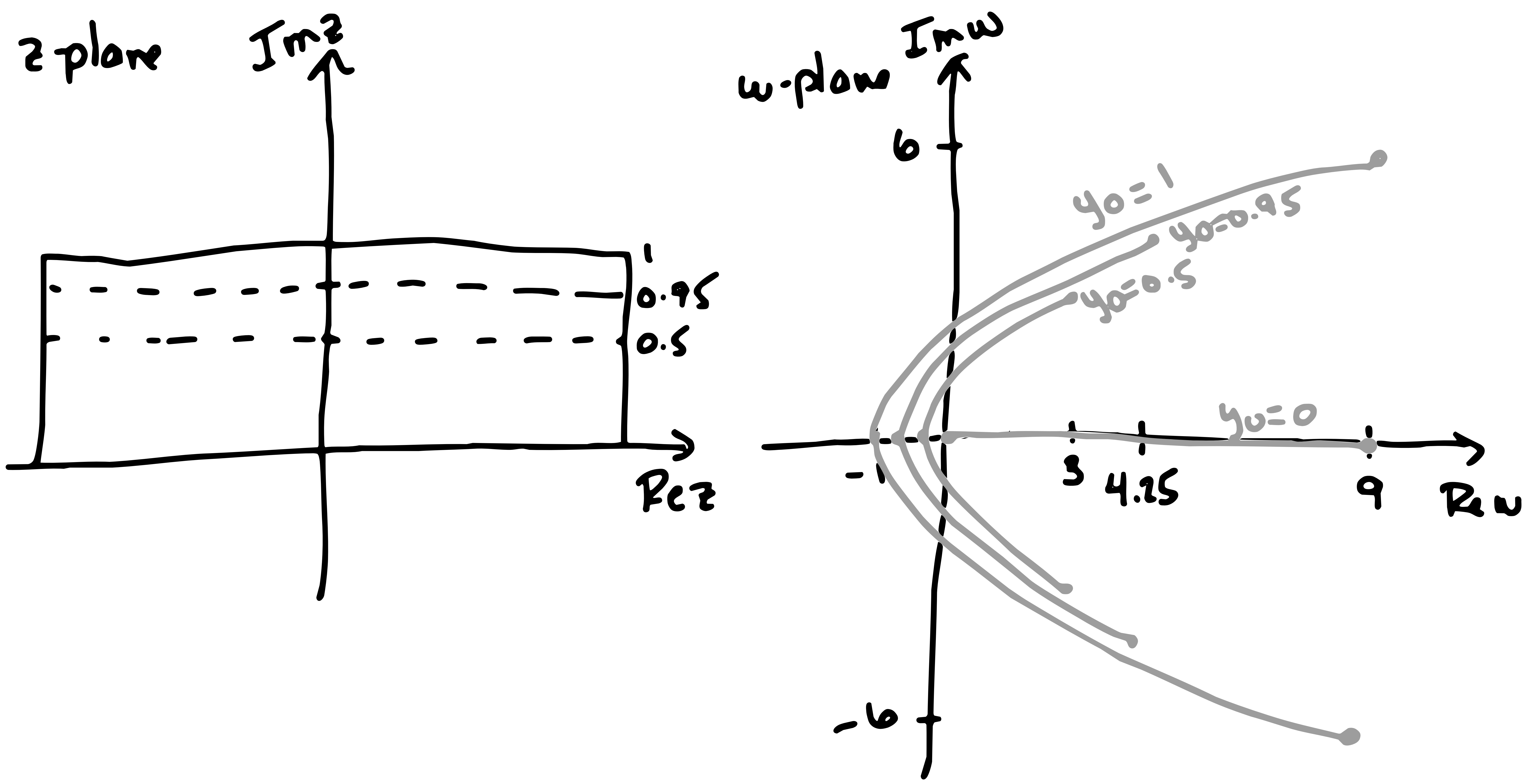
$$u = x^2 \quad \text{and} \quad v = 0 \qquad (y_0 = 0).$$

$$\Rightarrow \quad u = \frac{v^2}{4y_0^2} - y_0^2 \qquad -6y_0 \le v \le 6y_0$$

(parabola)

and $\quad 0 \le u \le 9 \quad$ (horizontal segment)

$v = 0$



Using the boundaries of the rectangle only, we obtain the following: