Section 1.5 M444

Problems Solution Pierre Parisé

Problem 1

We have $|a_n| = \sqrt{0^2 + \frac{\sin^2(n\,\pi/2)}{n^2}}$

$$= \frac{|\sin(n\,\pi/2)|}{n} \leq \frac{1}{n}$$

because $|\sin\theta| \leq 1$ for any angle θ.

Since $\frac{1}{n} \to 0$, so do $|a_n|$. Since $|a_n| \to 0$, so does a_n. Hence

$$a_n \to 0.$$

Problem 3

We have $|n+i| = \sqrt{n^2+1}$

$$\Rightarrow \quad |a_n| = \frac{1}{|n+i|} = \frac{1}{\sqrt{n^2+1}}.$$

Since $n^2+1 \geq n^2$ $\forall n \geq 1$ $\Rightarrow$ $\frac{1}{n^2+1} \leq \frac{1}{n^2}$

$$\Rightarrow \frac{1}{\sqrt{n^2+1}} \leq \frac{1}{n}.$$

Thus, $|a_n| = \frac{1}{\sqrt{n^2+1}} \leq \frac{1}{n}$.

Since $\frac{1}{n} \to 0$, so does $|a_n|$.

Since $|a_n| \to 0$, so does a_n.

Hence, $a_n \to 0$.

Problem 6

We will do some arithmetic.

$$a_n = \frac{(1+2i)n^2 + 2n - 1}{3in^2 + i}$$

$$= \frac{n^2 + 2in^2 + 2n - 1}{(3n^2+1)i} \cdot \frac{\bar{i}}{\bar{i}}$$

$$= \frac{-i(n^2 + 2n - 1 + 2n^2 i)}{3n^2+1}$$

$$= \frac{2n^2 + (1 - 2n - n^2)i}{3n^2+1}$$

Thus, $\text{Re } a_n = x_n = \frac{2n^2}{3n^2+1}$ and

$$\operatorname{Im} a_n = y_n = \frac{1-2n-n^2}{3n^2+1} .$$

Limit $\operatorname{Re} a_n$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{2n^2}{3n^2+1}$$

$$= \lim_{n\to\infty} \frac{2}{3+1/n^2}$$

$$= \frac{2}{3+0} \qquad \left(\lim_{n\to\infty} \frac{1}{n^2} = 0 \right)$$

$$\Rightarrow \lim_{n\to\infty} x_n = \frac{3}{2} .$$

Limit $\operatorname{Im} a_n$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{1-2n-n^2}{3n^2+1}$$

$$= \lim_{n\to\infty} \frac{1/n^2 - 2/n - 1}{3+1/n^2}$$

$$= \frac{0-0-1}{3} \Rightarrow \lim_{n\to\infty} y_n = \frac{-1}{3}$$

Hence, by Thm 1.5.8,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n$$

$$= \boxed{\frac{3}{2} - \frac{i}{3}}$$

Problem 11

Here $a_n = \dfrac{\cos\left(\frac{n\pi}{2}\right) + i\sin\left(\frac{n\pi}{2}\right)}{3^n}$.

We have

$$|a_n| = \sqrt{\frac{\cos^2\left(\frac{n\pi}{2}\right) + \sin^2\left(\frac{n\pi}{2}\right)}{3^{2n}}} = \frac{1}{3^n}.$$

The series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n}$ converges. So, by the comparison test for series,

$$\sum_{n=0}^{\infty} \frac{\cos\left(\frac{n\pi}{2}\right) + i\sin\left(\frac{n\pi}{2}\right)}{3^n}$$

converges. Now, notice that

$$\cos\left(\frac{n\pi}{2}\right) + i\sin\left(\frac{n\pi}{2}\right) = i^n, \quad \forall n \geq 0$$

Thus,

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{i^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{i}{3}\right)^n .$$

Since $\left|\frac{i}{3}\right| = \frac{1}{3} < 1$, the sum is

$$\frac{1}{1 - i/3} = \frac{3}{3-i} .$$

Hence

$$\sum_{n=0}^{\infty} \frac{\cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}}{3^n} = \boxed{\frac{3}{3-i}} .$$

Problem 12

Notice that $\left|\frac{1+i}{2}\right| = \frac{\sqrt{2}}{2} < 1$. By the comparison test with the geometric series $\sum_{n=0}^{\infty} \left(\frac{\sqrt{2}}{2}\right)^n$, the series $\sum_{n=0}^{\infty} \left(\frac{1+i}{2}\right)^n$ is convergent. Now, its sum is

$$\frac{1}{1-\frac{i+1}{2}} = \boxed{\frac{2}{1-i}}$$

Problem 18

We have

$$a_n = \frac{n^2}{(n+i)(n+200+2i)} = \frac{n^2}{n^2+200n-2+(3n+200)i}$$

and so

$$\lim_{n\to\infty} a_n = \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty} 1+\frac{200}{n}-\frac{2}{n^2}+\left(\frac{3}{n}+\frac{200}{n^2}\right)i}$$

$$= \frac{1}{1}$$

$$\Rightarrow \lim_{n\to\infty} a_n = 1 \neq 0.$$

Hence, by Thm. 1.5.17, $\sum_{n=0}^{\infty} a_n$ diverges.

Problem 22

n-th term is:

$$(-1)^n \frac{2^n+4^n}{(1+3i)^n} = \left(\frac{-2}{1+3i}\right)^n + \left(\frac{-4}{1+3i}\right)^n .$$

If we show that

$$\sum_{n=1}^{\infty} \left(\frac{-2}{1+3i}\right)^n \quad \text{and} \quad \sum_{n=1}^{\infty} \left(\frac{-4}{1+3i}\right)^n$$

converge, then we would be done.

We have:

$$\left|\frac{-2}{1+3i}\right| = \frac{2}{\sqrt{10}} \leq \frac{2}{\sqrt{9}} = \frac{2}{3} < 1.$$

This is a convergent geometric series.

We also have

$$\left|\frac{-4}{1+3i}\right| = \frac{4}{\sqrt{10}} > \frac{4}{\sqrt{16}} = 1 .$$

This is a divergent geometric series.

We can't conclude anything. But,

if $a_n = (-1)^n \dfrac{2^n + 4^n}{(1+3i)^n}$ and

$b_n = \left(\dfrac{-2}{1+3i}\right)^n$ and $c_n = \left(\dfrac{-4}{1+3i}\right)^n$, and

we assume that $\sum a_n$ converges,

then

$$\sum a_n - \sum b_n = \sum (a_n - b_n)$$

converges because $\sum a_n$ & $\sum b_n$

do converge. But

$$a_n - b_n = c_n = \left(\frac{-4}{1+3i}\right)^n$$

and we saw that $\sum c_n$ diverges.

A contradiction and thus

$$\sum_{n=1}^{\infty} (-1)^n \frac{2^n + 4^n}{(1+3i)^n}$$

is a divergent series.

Problem 32

Let $a_n = \dfrac{1}{3+i^n}$, $n \geq 1$.

Notice that

$$|3+i^n| \leq |3| + |i^n| = 3+1 = 4$$

$$\Rightarrow \quad \frac{1}{4} \leq \frac{1}{|3+i^n|} = |a_n|.$$

Hence (a_n) can't converge to 0.

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{3+i^n} \text{ diverges.} \qquad \square$$

Problem 42

Assume $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Then $\sum_{n=1}^{\infty} a_n$ is convergent. Let S_N be the N-th partial sum of $\sum a_n$ for $N \geq 1$. Then

$$(*) \quad |S_N| = |a_1 + a_2 + \dots + a_N| \leq |a_1| + |a_2| + \dots + |a_N|$$

by the triangle inequality.

Since $\lim_{N\to\infty} s_N = \sum_{n=1}^{\infty} a_n$, then

$$\lim_{N\to\infty} |s_N| = \left|\sum_{n=1}^{\infty} a_n\right|.$$

Take $\lim_{N\to\infty}$ on both sides of (*)

$$\Rightarrow \lim_{N\to\infty} |s_N| \le \lim_{N\to\infty}\left(|a_1| + \dots + |a_N|\right)$$

$$\Rightarrow \left|\sum_{n=1}^{\infty} a_n\right| \le \sum_{n=1}^{\infty} |a_n|$$

$\square$