Section 1.6 M444

Problems Solution Pierre Parisé

Problem 10 $z_1 = 1+i$, $z_2 = 1-i$, $z_3 = 2+5i$.

(a) $e^{z_1} e^{z_2} e^{z_3} = e^{1+i} e^{1-i} e^{2+5i}$

$= e^{4+5i}$

$= e^4 (\cos 5 + i \sin 5)$

$= \boxed{e^4 \cos 5 + i e^4 \sin 5}$

(b) $\dfrac{1}{e^{z_1}} = \dfrac{1}{e^{1+i}} = e^{-1-i} = e^{-1}(\cos(-1) + i\sin(-1))$

$= \boxed{e^{-1}\cos(1) - e^{-1}\sin(1)}$

(c) $(e^{z_1} e^{z_2})^{10} = (e^{z_1+z_2})^{10} = e^{10z_1 + 10z_2}$

$= e^{10+10i + 10-10i} = \boxed{e^{20}}$

(d) $\dfrac{e^{z_1} + e^{z_2}}{e^{z_3}} = \dfrac{e^1(\cos 1 + i\sin 1) + e^1(\cos(1) - i\sin(1))}{e^2 e^{+5i}}$

$= \dfrac{2e^1\cos(1)}{e^2 e^{5i}} = 2e^{-1} e^{-5i}$

$= \boxed{2e^{-1}(\cos 5 - i \sin 5)}$

Problem 15 b

Let $z = x+iy$. Then

$$e^{z^2} = e^{(x+iy)^2}.$$

Now, $(x+iy)^2 = x^2-y^2 + 2xyi$

$$\Rightarrow \quad e^{z^2} = e^{x^2-y^2}\, e^{i2xy}$$

$$= e^{x^2-y^2}\cos 2xy + i\, e^{x^2-y^2}\sin 2xy$$

Thus,

$$u(x,y) = e^{x^2-y^2}\cos(2xy)$$

and

$$v(x,y) = e^{x^2-y^2}\sin(2xy).$$

Problem 16b

Let $z = x+iy$ so that $\bar{z} = x+iy$. We have

$$\overline{e^{z}} = \overline{e^{x} e^{iy}} = \overline{e^{x}}\,\overline{e^{iy}} = e^{x} e^{-iy} = e^{x-iy} = e^{\bar{z}}. \quad \square$$

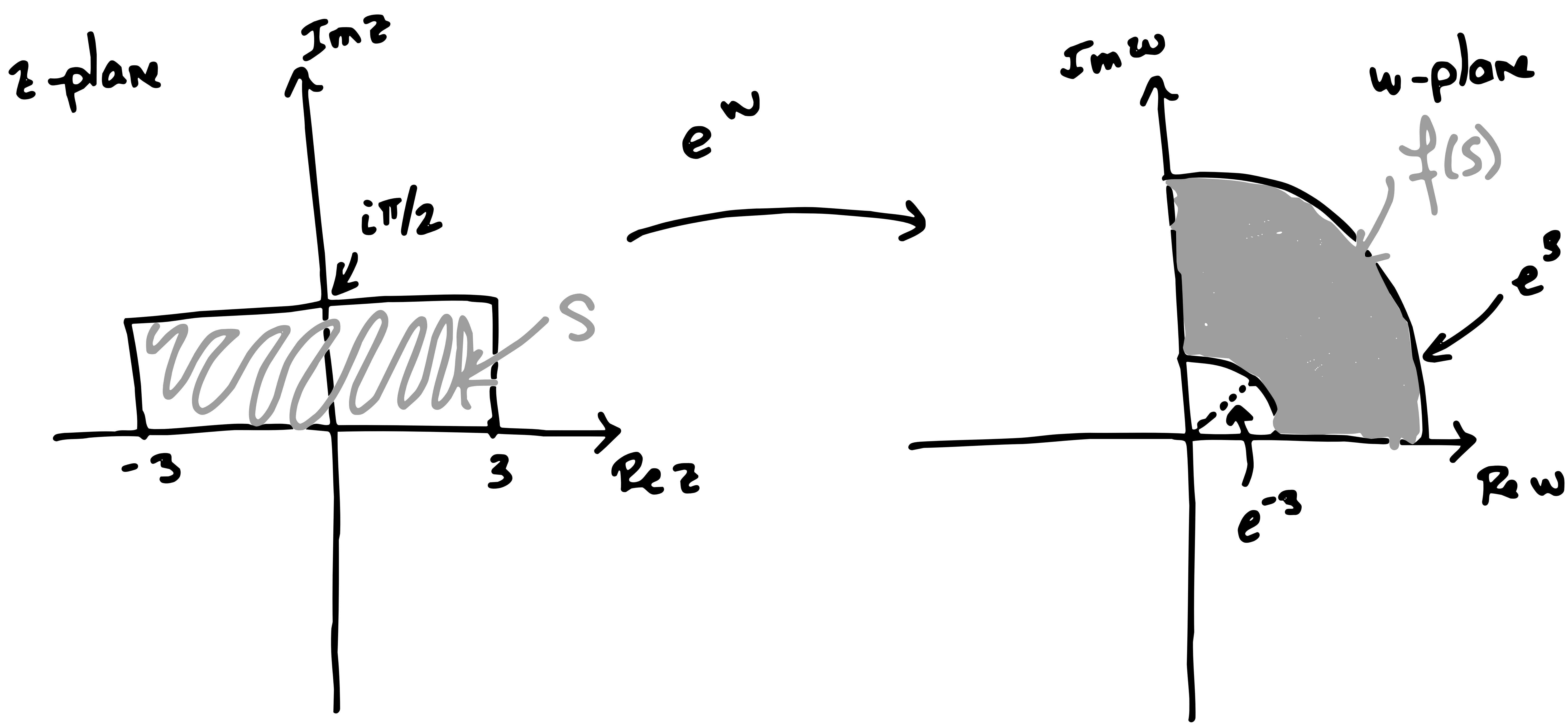
Problem 17

Let $S = \{ x+iy \in \mathbb{C} : -3 \le x \le 3,\ 0 \le y \le \pi/2 \}$.

Recall that

$$e^{z} = e^{x+iy} = e^{x} e^{iy}.$$

$$\Rightarrow \quad |e^{z}| = e^{x} \quad \& \quad \mathrm{Arg}(e^{z}) = y.$$



$$-3 \le \mathrm{Re}\, z \le 3 \quad \Rightarrow \quad e^{-3} \le |e^{z}| \le e^{3}$$

$$0 \le \mathrm{Im}\, z \le \pi/2 \quad \Rightarrow \quad 0 \le \mathrm{Arg}(e^{z}) \le \pi/2$$

Problem 28

(a) No. $e^{i} = e^{(1+2\pi)i}$, but $i \neq (1+2\pi)i$.

So, e^{z} is not one-to-one (injective!).

(b) No. Let $z_1 = 1 + i$ and $z_2 = 2i$

$$\Rightarrow \quad |z_1| = \sqrt{2} < 2 = |z_2|$$

but $|e^{z_1}| = e > e^{0} = |e^{z_2}|$.

(c) No. Using the fact $\omega \neq 0 \Leftrightarrow |\omega| \neq 0$,

we see that

$$e^{z} \neq 0 \Leftrightarrow |e^{z}| \neq 0 \Leftrightarrow e^{x} \neq 0.$$

We know that $e^{x} \neq 0$, $\forall x$

$$\Rightarrow \quad e^{z} \neq 0 \quad , \forall z \in \mathbb{C}.$$

(d) No. We have

$$e^{i\pi/4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i \notin \mathbb{R}^{+}$$

(e) We have $|e^{z}| = e^{x}$, not e^{z}.

(f) No. $e^{z} = 1 \Leftrightarrow z = 2k\pi i$, $k \in \mathbb{Z}$.