Section 1.7 M444

Problems Solution Pierre Parisé

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## Problem 2

a) Set $z = \frac{\pi}{4} + i\frac{\pi}{4}$.

Then $e^{iz} = e^{i\pi/4} e^{-\pi/4} = e^{-\pi/4}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$

and $e^{-iz} = e^{-i\pi/4} e^{\pi/4} = e^{\pi/4}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$.

Hence,

$$\cos\left(\frac{\pi}{4} + i\frac{\pi}{4}\right) = \frac{e^{-\pi/4}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) + e^{\pi/4}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)}{2}$$

$$= \frac{\sqrt{2}}{2}\left(\frac{e^{\pi/4} + e^{-\pi/4}}{2}\right) - i\frac{\sqrt{2}}{2}\left(\frac{e^{\pi/4} - e^{-\pi/4}}{2}\right)$$

$$\approx 0.9366 - 0.6142\, i$$

$$\sin\left(\frac{\pi}{4} + i\frac{\pi}{4}\right) = \left[e^{-\pi/4}\left(\sqrt{2}/2 + i\frac{\sqrt{2}}{2}\right) - e^{\pi/4}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)\right]/2i$$

$$= \frac{\sqrt{2}}{2}\left(\frac{e^{-\pi/4} - e^{\pi/4}}{2i}\right) + i\frac{\sqrt{2}}{2}\left(\frac{e^{\pi/4} + e^{-\pi/4}}{2i}\right)$$

$$\approx 0.9366 - 0.6142\, i.$$

b) Set $z = -i\frac{\pi}{4}$. then

$$e^{iz} = e^{\pi/4} \quad \text{and} \quad e^{-iz} = e^{-\pi/4}.$$

Hence

$$\cos\left(\frac{-i\pi}{4}\right) = \frac{e^{\pi/4} + e^{-\pi/4}}{2} = \cosh(\pi/4)$$

$$\approx 1.3246$$

and

$$\sin\left(\frac{-i\pi}{4}\right) = \frac{e^{\pi/4} - e^{-\pi/4}}{2i} = \frac{\sinh(\pi/4)}{i}$$

$$= -1.9653i.$$

## Problem 3

(a) Let $z = x+iy$, so that $\bar{z} = x - iy$. Then

$$\cos z = \frac{e^{-y}e^{ix} + e^{y}e^{-ix}}{2}$$

$$\Rightarrow \quad \cos\bar{z} = \frac{e^{y}e^{ix} + e^{-y}e^{-ix}}{2}$$

and

$$\overline{\cos z} = \frac{\overline{e^{-y}e^{ix}} + \overline{e^{y}e^{-ix}}}{2}$$

$$= \frac{e^{-y}e^{-ix} + e^{y}e^{ix}}{2} = \cos\bar{z}.$$

(b) Similarly, we have

$$\sin z = \frac{e^{-y}e^{ix} - e^{y}e^{-ix}}{2i}$$

So,

$$\sin \bar{z} = \frac{e^{y}e^{ix} - e^{-y}e^{-ix}}{2i}$$

and

$$\overline{\sin z} = \frac{\overline{e^{-y}e^{ix}} - \overline{e^{y}e^{-ix}}}{\overline{2i}}$$

$$= \frac{e^{-y}e^{-ix} - e^{y}e^{ix}}{-2i}$$

$$= \frac{e^{y}e^{ix} - e^{-y}e^{-ix}}{2i} = \sin \bar{z}$$

**Problem 10**

$\cos(z^2) = \dfrac{e^{iz^2} + e^{-iz^2}}{2}$. Let $z = x+iy$, so that

$$z^2 = x^2 - y^2 + 2xyi .$$

Then $$e^{iz^2} = e^{i(x^2-y^2)} e^{-2xy}$$

and $$e^{-iz^2} = e^{-i(x^2-y^2)} e^{2xy}$$

Hence,

$$\cos(z^2) = \frac{\left(e^{-2xy}\left(\cos(x^2-y^2) + i\sin(x^2-y^2)\right) + e^{2xy}\left(\cos(x^2-y^2) - i\sin(x^2-y^2)\right)\right)}{2}$$

$$= \frac{\cos(x^2-y^2)\left(e^{2xy} + e^{-2xy}\right)}{2} \quad \swarrow \text{Re.}$$

$$+ i\,\frac{\sin(x^2-y^2)\left(e^{-2xy} - e^{2xy}\right)}{2} \quad \swarrow \text{Im.}$$

## Problem 24

Assume $\sinh z = 0$

$$\Leftrightarrow \frac{e^z - e^{-z}}{2} = 0$$

$$\Leftrightarrow e^z = e^{-z} \Leftrightarrow e^x e^{iy} = e^{-x} e^{-iy}$$

So, $e^{2x} = 1$ and $e^{2iy} = 1$

$\Leftrightarrow$ $x = 0$ and $2y = 2k\pi$, $k \in \mathbb{Z}$

$\Leftrightarrow$ $z = ik\pi$, $k \in \mathbb{Z}$.

Now, $\cosh z = 0$

$$\Leftrightarrow \quad \frac{e^{z} + e^{-z}}{2} = 0$$

$$\Leftrightarrow \quad e^{x} e^{iy} = -e^{-x} e^{-iy}$$

So,

$$e^{2x} = 1 \quad \text{and} \quad e^{2iy} = -1$$

$$\Leftrightarrow \quad x = 0 \quad \text{and} \quad 2y = \pi + 2k\pi, \ k \in \mathbb{Z}$$

$$\Leftrightarrow \quad z = i \, \frac{(2k+1)}{2} \pi \quad , \ k \in \mathbb{Z} .$$

## Problem 29

$$\sin(z_1 + z_2) = \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{2i}$$

$$\sin(z_1)\cos(z_2) = \left( \frac{e^{iz_1} - e^{-iz_1}}{2i} \right) \left( \frac{e^{iz_2} + e^{-iz_2}}{2} \right)$$

$$= \frac{e^{i(z_1+z_2)} + e^{i(z_1-z_2)} - e^{i(z_2-z_1)} - e^{-i(z_1+z_2)}}{4i}$$

Similarly

$$\cos(z_1)\sin(z_2) = \frac{e^{i(z_1+z_2)} - e^{i(z_1-z_2)} + e^{i(z_2-z_1)} - e^{-i(z_1+z_2)}}{4i}$$

Hence

$$\sin(z_1)\cos(z_2) + \sin(z_2)\cos(z_1)$$

$$= \frac{2e^{i(z_1+z_2)} - 2e^{-i(z_1+z_2)}}{4i}$$

$$= \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{2i} = \sin(z_1+z_2). \quad \square$$

Problem 40.

$$\cosh^2(z) - \sinh^2(z) = \left(\frac{e^{z}+e^{-z}}{2}\right)^2 - \left(\frac{e^{z}-e^{-z}}{2}\right)^2$$

$$= \frac{\cancel{e^{2z}} + 2 + \cancel{e^{-2z}} \cancel{- e^{2z}} + 2 - \cancel{e^{-2z}}}{4}$$

$$= \frac{4}{4}$$

$$= 1. \quad \square$$