Section 1.8 M444

Problems Solution Pierre Parisé

Problem 4

(a) $\log z = \log|z| + i\,(\mathrm{Arg}(z) + 2\pi i)$.

So, $|z| = \dfrac{1}{(\sqrt{2})^4} = \dfrac{1}{4}$.

The argument of z is a little more delicate. We have

$$\arg(1+i) = \left\{ \pi/4 + 2k\pi i \;:\; k \in \mathbb{Z} \right\}.$$

so

$$\arg\left((1+i)^4\right) = 4\arg(1+i) = \left\{ \pi + 2k\pi i : k \in \mathbb{Z} \right\}$$

so

$$\arg\left(\frac{1}{(1+i)^4}\right) = -\arg(1+i)^4 = \left\{ -\pi + 2k\pi i : k \in \mathbb{Z} \right\}.$$

So, $\mathrm{Arg}\left(\dfrac{1}{(1+i)^4}\right) = \pi$

$$\Rightarrow \log\left(\frac{1}{(1+i)^4}\right) = \log\left(\frac{1}{4}\right) + (\pi + 2k\pi)i \;, k \in \mathbb{Z}.$$

(b) $|z| = (\sqrt{2})^{11} = (\sqrt{2})^{10}\sqrt{2} = 32\sqrt{2}$.

Also

$$\arg z = 11\arg(1-i) = \left\{-\frac{11\pi}{4} + 2k\pi : k \in \mathbb{Z}\right\}.$$

$$\Rightarrow \quad \operatorname{Arg} z = -\frac{11\pi}{4} + 2k = -\frac{3\pi}{4}.$$

$$\Rightarrow \log(z) = \log(32\sqrt{2}) + \left(-\frac{3\pi}{4} + 2k\pi\right)i$$

(c) $|z| = 4^8 = 65536 = 2^{16}$

Also,

$$\arg z = 8\arg(1+i\sqrt{3}) = \left\{\frac{8\pi}{3} + 2k\pi : k \in \mathbb{Z}\right\}.$$

$$\Rightarrow \quad \operatorname{Arg} z = \frac{8\pi}{3} - 2\pi = \frac{2\pi}{3}$$

$$\Rightarrow \log(z) = 16\log(2) + \left(\frac{2\pi}{3} + 2k\pi\right)i \quad , k \in \mathbb{Z}.$$

(d)

$$\frac{e^{-i\pi/7}}{2e^{-i\pi/5}} = \frac{1}{2}e^{i(-\pi/7 + \pi/5)}$$

$$= \frac{1}{2}e^{i\left(\frac{2\pi}{35}\right)}.$$

$$\Rightarrow \log(z) = \log\left(\frac{1}{2}\right) + \left(\frac{2\pi}{35} + 2k\pi\right)i \quad , k \in \mathbb{Z}.$$

Problem 6

(a) $\mathrm{Log}(z) = \log\left(\frac{1}{4}\right) + \pi i$.

(b) $\mathrm{Log}(z) = \log\left(32\sqrt{2}\right) - \frac{3\pi}{4} i$.

(c) $\mathrm{Log}(z) = 16\log(2) + \frac{2\pi}{3} i$.

(d) $\mathrm{Log}(z) = \log\left(\frac{1}{2}\right) + \frac{2\pi}{35} i$.

Problem 14

$$e^{-z} = 1 + i \iff \frac{1}{1+i} = e^{z}$$

Thus,
$$z = \mathrm{Log}\left(\frac{1}{1+i}\right) + 2k\pi i \quad , k \in \mathbb{Z}.$$

So,
$$\arg\left(\frac{1}{1+i}\right) = \left\{-\frac{\pi}{4} + 2k\pi : k \in \mathbb{Z}\right\}.$$

Thus,
$$z = \log\left(\frac{1}{\sqrt{2}}\right) + \frac{-\pi}{4} i + 2k\pi i \quad , k \in \mathbb{Z}.$$

Problem 18

$$e^z = \frac{1+i}{1-i} \iff e^z = i$$

Thus, $z = \operatorname{Log}(i) + 2k\pi i \ , \ k \in \mathbb{Z}$.

$$\Rightarrow z = \log(1) + \frac{\pi}{2} i + 2k\pi i \ , \ k \in \mathbb{Z}.$$

Problem 19

(a) $\operatorname{Log}(e^{i\pi}) = \log(1) + \pi i \quad (\operatorname{Arg}(e^{i\pi}) = \pi)$.

$= \pi i$

$\operatorname{Log}(e^{3i\pi}) = \log(1) + \pi i = \pi i \quad (\operatorname{Arg}(e^{i3\pi}) = \pi)$

$\operatorname{Log}(e^{5i\pi}) = \log 1 + \pi i = \pi i \quad (\operatorname{Arg}(e^{i5\pi}) = \pi)$.

(b) Let $z = x + iy$, so that $|e^z| = e^x$. Then

$$\operatorname{Log}(z) = z \iff \log|e^z| + i\operatorname{Arg}(e^z) = x + iy$$

$$\iff \log e^x + i\operatorname{Arg}(e^z) = x + iy$$

$$\iff \operatorname{Arg}(e^z) = y \iff -\pi < y \leq \pi. \quad \square$$

Problem 20

Assume that $\log z = i\arg(z)$. Then, there is a $k \in \mathbb{Z}$ and $m \in \mathbb{Z}$ s.t.

$$\log|z| + (\operatorname{Arg}(z) + 2k\pi)i = i(\operatorname{Arg}(z) + 2m\pi)$$

Comparing complex number, we obtain

$$\log|z| = 0 \Rightarrow |z| = 1.$$

Now, let $|z| = 1$. Then,

$$\log z = \log|z| + i(\operatorname{Arg} z + 2k\pi)$$
$$= i(\operatorname{Arg} z + 2k\pi) = i \arg z. \quad \square$$

Problem 30

$$(1+i)^{3+i} = e^{(3+i)\operatorname{Log}(1+i)}$$

So, $\operatorname{Log}(1+i) = \log\sqrt{2} + \frac{\pi}{4}i$

$$\Rightarrow (3+i)\operatorname{Log}(1+i) = 3\log\sqrt{2} - \frac{\pi}{4} + \left(\log\sqrt{2} + \frac{3\pi}{4}\right)i$$

$$\Rightarrow (1+i)^{3+i} = e^{3\log\sqrt{2} - \frac{\pi}{4} + \left(\log\sqrt{2} + \frac{3\pi}{4}\right)i}$$

$$= e^{3\log\sqrt{2} - \pi/4}\left(\cos\left(\log\sqrt{2} + \frac{3\pi}{4}\right) + i\sin\left(\log\sqrt{2} + \frac{3\pi}{4}\right)\right)$$

$$\approx -0.9412 + 0.4417 .$$

Problem 31

$$i^i = e^{i \operatorname{Log} i}$$

Here, $\operatorname{Log} i = \frac{\pi}{2} i \Rightarrow i^i = e^{i\left(\frac{\pi}{4} i\right)} = e^{-\pi/4}$

$$\Rightarrow i^i \approx 0.4559 .$$