Section 2.2 M444

Problem Solutions P.-O. Parisé

Problem 1

$$\lim_{z\to i}\left(3z^2+3z-1\right) = \lim_{z\to i} 3z^2 + \lim_{z\to i} 3z + \lim_{z\to i}(-1)$$

[sum of limits]

$$= 3\lim_{z\to i} z^2 + 3\lim_{z\to i} z - \lim_{z\to i} 1$$

[product of limits]

$$= 3(i^2) + 3(i) - 1$$

$$= (-1) - 1 + 3i = -2+3i.$$

Problem 4

$$\lim_{z\to 2}\frac{z^4-16}{z-2} = \lim_{z\to 2}\frac{(z^2-4)(z^2+4)}{z-2}$$

$$= \lim_{z\to 2}\frac{\cancel{(z-2)}(z+2)(z^2+4)}{\cancel{z-2}}$$ [$z \neq 2$ in a limit]

$$= \lim_{z\to 2}(z+2)(z^2+4)$$

$$= \left(\lim_{z\to 2} z+2\right)\left(\lim_{z\to 2} z^2+4\right)$$ [Product]

$$= \left(\lim_{z \to 2} z + 2\right)\left(\lim_{z \to 2} z^2 + 4\right)$$

$$= (2+2)(4+4) = \boxed{32}$$

Problem 6

Here, we use the Squeeze Theorem because $\lim_{z \to 0} \operatorname{Arg}(z)$ do not exist.

We have that $\operatorname{Arg}(z) \in (-\pi, \pi]$

$$\Rightarrow \quad |\operatorname{Arg}(z)| \leq \pi$$

Hence

$$0 \leq |z \operatorname{Arg}(z)| \leq |z|\,|\operatorname{Arg}(z)|$$

$$\leq \pi\,|z|.$$

Squeeze Theorem then implies that

$$\lim_{z \to 0} |z \operatorname{Arg}(z)| = 0.$$

Problem 9 WTS: $\lim_{z \to -3} (\text{Arg } z)^2 = \pi^2$.

Notice that

$$|\text{Arg}(z)^2 - \pi^2| = |\text{Arg } z - \pi|\,|\text{Arg } z + \pi|.$$

Choose $\delta_1 > 0$ s.t.

$$0 < |z+3| < \delta_1 \;\&\; \text{Im } z \geq 0 \Rightarrow |\text{Arg } z - \pi| < \frac{\varepsilon}{2\pi}$$

Choose $\delta_2 > 0$ s.t.

$$0 < |z+3| < \delta_2 \;\&\; \text{Im } z < 0 \Rightarrow |\text{Arg } z + \pi| < \frac{\varepsilon}{2\pi}$$

Let $\delta := \min\{\delta_1, \delta_2\}$. Let $0 < |z - z_0| < \delta$.

If $\text{Im } z \geq 0$, then

$$|[\text{Arg}(z)]^2 - \pi^2| \leq \frac{\varepsilon}{2\pi} 2\pi = \varepsilon.$$

Similarly fn $\text{Im } z < 0$.

Problem 11

We have

$$\sin \bar{z} = \sin(x)\cosh(-y) + i\cos(x)\sinh(-y)$$
$$= \sin(x)\cos(y) - i\cos(x)\sinh(y).$$

Hence

$$\lim_{(x,y)\to(0,0)} \sin(x)\cos(y) = \lim_{x \to 0} \sin x \lim_{y \to 0} \cos y$$
$$= (0)(1) = 0$$

and

$$\lim_{(x,y)\to(0,0)} -\cos(x)\sinh(y) = -\lim_{x\to 0}\cos(x)\lim_{y\to 0}\sinh(y)$$

$$= -(1)(0) = 0$$

Hence

$$\lim_{z\to 0}\sin(\bar{z}) = 0 + 0i = 0.$$

Problem 12

Notice that for $z \neq 0$,

$$\left|e^{i/|z|^2}\right| = 1$$

Hence by the Squeeze Theorem:

$$\lim_{z\to 0} z\, e^{i/|z|^2} = \lim_{z\to 0} z = 0.$$

Problem 13

$$\lim_{z\to\infty}\frac{z+1}{3iz+2} = \lim_{z\to\infty}\frac{1+1/z}{3i+2/z}$$

$$= \lim_{z\to\infty} 1 + \lim_{z\to\infty} 1/z \Big/ \lim_{z\to 0} 3i + \lim_{z\to 0}\frac{2}{z}$$

$$= \frac{1+0}{3i+0} = \boxed{\frac{1}{3i}}.$$

Problem 15

$$\lim_{z\to\infty}\left(\frac{z^3+i}{z^3-i}\right)^2 = \lim_{z\to\infty}\left(\frac{1+i/z^3}{1-i/z^3}\right)^2$$

$$= \frac{\left[\lim_{z\to\infty}(1+i/z^3)\right]\left[\lim_{z\to\infty} 1+ i/z^3\right]}{\left[\lim_{z\to\infty} 1-i/z^3\right]\left[\lim_{z\to\infty} 1-i/z^3\right]}$$

$$= \frac{(1+0)(1+0)}{(1-0)(1-0)} = \boxed{1}$$

Problem 18

We have $\quad \mathrm{Log}\, z = \log|z| + i\,\mathrm{Arg}(z).$

Since $|\mathrm{Arg}(z)| \le \pi$ and $\lim_{z\to\infty}\frac{1}{|z|} = 0$,

then by the Squeeze Theorem

$$\lim_{z\to\infty}\frac{1}{z}\mathrm{Arg}(z) = 0 . \quad (*)$$

For any $u > 0$, we have the following

inequality:

$$\log u \le u - 1 .$$

Hence, for $|z| \ge 1$,

$$\log |z|^{1/2} \le |z|^{1/2} - 1 .$$

Applying this to $\log|z|$, we get

$$\log|z| = 2\log|z|^{1/2} \le 2|z|^{1/2} - 2$$

$$\Rightarrow \quad \frac{\log|z|}{|z|} \le \frac{2}{|z|^{1/2}} - \frac{2}{|z|}, \quad |z| \ge 1$$

It is straightforward to see that

$$\lim_{z\to\infty} \frac{1}{|z|^{1/2}} = 0 .$$

Hence by the Squeeze Theorem,

$$\lim_{z\to\infty} \frac{\log|z|}{|z|} = 0 . \qquad (**)$$

Combining $(*)$ with $(**)$, we get

$$\lim_{z\to\infty} \frac{\operatorname{Log}(z)}{z} = 0 .$$

Problem 19

Let $z=-3+iy$, $y>0$. Then

$$\operatorname{Arg}(z) = \operatorname{Arg}(-3+iy) = \arctan\left(\frac{y}{-3}\right)+\pi$$

$$\Rightarrow \lim_{y\to 0^+} \operatorname{Arg}(z) = \lim_{y\to 0^+} \arctan\left(\frac{y}{-3}\right)+\pi$$

$$= \arctan(0)+\pi = \pi.$$

But, for $z=-3+iy$, $y<0$, then

$$\operatorname{Arg}(z) = \operatorname{Arg}(-3+iy) = \arctan\left(\frac{y}{-3}\right)-\pi$$

$$\Rightarrow \lim_{y\to 0^-} \operatorname{Arg}(z) = \lim_{y\to 0^-} \arctan\left(\frac{y}{-3}\right)-\pi$$

$$= \arctan(0)-\pi = -\pi.$$

Thus, two possible limits, a contradiction with the uniqueness of limits.

Thus, $\lim_{z\to -3} \operatorname{Arg}(z)$ does not exist.

Problem 23

Let $z = x > 0$. Then

$$\lim_{x \to 0^+} e^{1/x} = \lim_{x \to +\infty} e^{x} = +\infty$$

Let $z = -x$, $x > 0$. Then

$$\lim_{x \to 0^+} e^{-1/x} = \lim_{x \to +\infty} e^{-x} = 0.$$

The limit is not unique and therefore, $\lim_{z \to 0} e^{1/z}$ does not exist.

Problem 25

We have

$$\lim_{x \to 0^+} \frac{x}{|x|} = \lim_{x \to 0^+} \frac{x}{x} = 1.$$

But

$$\lim_{x \to 0^-} \frac{x}{|x|} = \lim_{x \to 0^-} -\frac{x}{x} = -1.$$

So, $\lim_{z \to 0} \frac{z}{|z|}$ does not exist.

Problem 26

Choose $z = iy$, $y > 0$.

$$\Rightarrow \lim_{z \to 0} \frac{\operatorname{Im} z}{z} = \lim_{y \to 0^+} \frac{y}{iy}$$

$$= \frac{1}{i}$$

Choose $z = x$, $x > 0$.

$$\Rightarrow \lim_{z \to 0} \frac{\operatorname{Im} z}{z} = \lim_{x \to 0^+} \frac{0}{x} = 0.$$

Hence, $\lim_{z \to 0} \frac{\operatorname{Im} z}{z}$ does not exist.

Problem 31

Assume $\exists B(z_0, r)$ s.t.

$$f(z) \neq 0, \quad \forall z \in B(z_0, r).$$

($\Rightarrow$) Assume also that $\lim_{z \to z_0} f(z) = 0$.

<u>WST</u>: $\lim_{z \to z_0} \frac{1}{f(z)} = \infty$.

Let $M > 0$. Then, from the assumption, $\exists \delta_1 > 0$ s.t. $0 < |z - z_0| < \delta_1 \Rightarrow |f(z)| < \varepsilon = \frac{1}{M}$.

Let $\delta := \min\{\delta_1, r\}$. Then, if

$$0 < |z - z_0| < \delta \quad \Rightarrow \quad \begin{cases} |f(z)| \neq 0 \\ |f(z)| < \frac{1}{M}. \end{cases}$$

Therefore, for $0 < |z - z_0| < \delta$,

$$|f(z)| < \frac{1}{M} \iff M < \frac{1}{|f(z)|}$$

$$\iff \left|\frac{1}{f(z)}\right| > M.$$

<u>Conclusion</u>: $\forall M > 0, \exists \delta > 0$ s.t.

$$0 < |z - z_0| < \delta \Rightarrow \left|\frac{1}{f(z)}\right| > M.$$

So, $\lim\limits_{z \to z_0} \frac{1}{f(z)} = \infty$.

($\Leftarrow$) Assume $\lim\limits_{z \to z_0} \frac{1}{f(z)} = \infty$.

<u>WST</u>: $\lim\limits_{z \to z_0} f(z) = 0$.

Let $\varepsilon > 0$. From our assumption, $\exists \delta_2 > 0$

such that $\delta_2 < r$ and

$$0 < |z - z_0| < \delta_2 \Rightarrow \left|\frac{1}{f(z)}\right| > M = \frac{1}{\varepsilon}.$$

Let $\delta := \delta_2$. Then, if $|z - z_0| < \delta$, then

$$\frac{1}{|f(z)|} > \frac{1}{\varepsilon}$$

$$\Leftrightarrow \quad \varepsilon > |f(z)|$$

$$\Leftrightarrow \quad |f(z)| < \varepsilon.$$

Conclusion: $\exists \delta > 0$ s.t.

$$0 < |z - z_0| < \delta \Rightarrow |f(z)| < \varepsilon.$$

Hence, $\lim_{z \to z_0} f(z) = 0$.

Problem 33

The function is discontinuous à $z = -1-3i$.
It is a rational function, so it is continuous on $\mathbb{C} \setminus \{-1-3i\}$.

Problem 35

We know from the limit properties that

$$\lim_{z \to z_0} \bar{z} = \bar{z}_0 \quad , \forall z_0 \in \mathbb{C}.$$

Hence, $\bar{z}$ is continuous on all of $\mathbb{C}$.

Problem 36

$f(z) = \text{Log}(z+1)$ is not defined at 0.

We also know that $\text{Log}(z)$ is discontinuous on $(-\infty, 0)$. Hence, $\text{Log}(1+z)$ is discontinuous on $(-\infty, -1)$.

We know that $\text{Log}(z)$ is continuous on $\mathbb{C} \setminus (-\infty, 0]$, so $\text{Log}(1+z)$ is continuous on $\mathbb{C} \setminus (-\infty, 1]$.

<u>Conclusion</u>: $\text{Log}(1+z)$ continuous on $\mathbb{C} \setminus (-\infty, -1]$ and discontinuous on $(-\infty, -1]$.