Problem 3

Let $z = x + iy$. Then,

$$e^{z^2} = e^{x^2-y^2+2xyi} = e^{x^2-y^2}\cos(2xy) + ie^{x^2-y^2}\sin(2xy).$$

Set $u(x,y) = e^{x^2-y^2}\cos(2xy)$ and $v(x,y) = e^{x^2-y^2}\sin(2xy)$.

We first have that

$$u_x = 2xe^{x^2-y^2}\cos(2xy) - 2ye^{x^2-y^2}\sin(2xy)$$

and

$$v_y = -2ye^{x^2-y^2}\sin(2xy) + 2xe^{x^2-y^2}\cos(2xy).$$

Hence $u_x = v_y$.

Also, we have

$$u_y = -2ye^{x^2-y^2}\cos(2xy) - 2xe^{x^2-y^2}\sin(2xy)$$

and

$$v_x = 2xe^{x^2-y^2}\sin(2xy) + 2ye^{x^2-y^2}\cos(2xy).$$

Hence, we have $u_y = -v_x$.

The function e^{z^2} satisfies the Cauchy-Riemann equations for any $z = x + iy \in \mathbb{C}$ and therefore it is analytic on $\mathbb{C}$.

Problem 5

Let $z = x + iy$, so that $\overline{z} = x - iy$. Then, we get

$$e^{\overline{z}} = e^x\cos(-y) + ie^x\sin(-y) = e^x\cos y - ie^x\sin y.$$

Set $u(x,y) = e^x\cos y$ and $v(x,y) = -e^x\sin y$.

We have

$$u_x = e^x\cos y \quad \text{and} \quad v_y = -e^x\cos y.$$

We see that $u_x \neq v_y$. Therefore, the function is nowhere analytic because the point z was arbitrary.

Problem 6

The function is defined on $\mathbb{C}\backslash\{0\}$. In this case, we have

$$u(x,y) = \frac{y}{x^2+y^2} \quad \text{and} \quad v(x,y) = -\frac{x}{x^2+y^2}.$$

Therefore, we compute

$$u_x = -\frac{2xy}{(x^2+y^2)^2} \quad \text{and} \quad v_y = \frac{2xy}{(x^2+y^2)^2}.$$

We see that $u_x \neq v_y$ when $x \neq 0$ and $y = 0$ or when $x = 0$ and $y \neq 0$.

However, we also have

$$u_y = \frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

and

$$v_x = \frac{-(x^2+y^2) + 2x^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}.$$

So, there are two cases to deal with:

① when $x = 0$ but $y \neq 0$, we obtain

$$u_y = -\frac{y^2}{(y^2)^2} = -\frac{1}{y^2}$$

and

$$v_x = -\frac{y^2}{(y^2)^2} = -\frac{1}{y^2}.$$

Therefore, $u_y \neq -v_x$ for any $y \neq 0$.

② When $x \neq 0$ but $y = 0$, we obtain

$$u_y = \frac{1}{x^2} \quad \text{and } v_x = \frac{1}{x^2}.$$

Therefore, $u_y \neq -v_x$.

In all the cases, the function does not satisfy the Cauchy-Riemann equations on its domain $\mathbb{C}\backslash\{0\}$. Therefore, it is analytic nowhere on its domain.

Problem 12

Let $z = x + iy$. Then, from Section 1.7, we have

$$\cosh(z) = \cosh(x)\cos(y) + i\sinh(x)\sin(y).$$

First, we compute

$$u_x = \sinh(x)\cos(y) \quad \text{and} \quad v_y = \sinh(x)\cos(y).$$

Therefore, $u_x = v_y$. Also, we compute

$$u_y = -\cosh(x)\sin(y) \quad \text{and} \quad v_x = \cosh(x)\sin(y).$$

Therefore, $u_y = -v_x$. Hence, the function $\cosh(z)$ satisfies the Cauchy-Riemann equations on its domain $\mathbb{C}$ and is therefore analytic on $\mathbb{C}$.

Problem 33

Assume that $f = u + iv$ is analytic on a region Ω. There are two cases to deal with.

① Assume that $u(z) = c$ for any $z \in \Omega$, where c is a constant. Then, $u_x = 0$ and $u_y = 0$. Since f is analytic, it satisfies the C-R equations and therefore

$$v_y = u_x = 0 \quad \text{and} \quad v_x = -u_y = 0.$$

Fix $w \in \Omega$. Since Ω is a region there is a polygonal curve C joining w to any other point $z \in \Omega$. By the Fundamental Theorem for line integrals, we have

$$\int_C \vec{\nabla} v \cdot d\vec{r} = v(z) - v(w).$$

But, $\vec{\nabla} v = \vec{0}$ and hence $v(z) - v(w) = 0$ for any $z \in \Omega$. In other words, we get $v(z) = v(w)$ for any $z \in \Omega$, where $v(w)$ is constant. Therefore, we get

$$f(z) = u(z) + iv(z) = c + iv(w) \quad \forall z \in \Omega.$$

This implies that f is constant.

② Assume now that $v(z) = c$ for any $z \in \Omega$, where c is a constant. Then, $v_x = v_y = 0$. Since f is analytic, it satisfies the C-R equations and therefore

$$u_x = v_y = 0 \quad \text{and} \quad u_y = -v_x = 0.$$

Fix $w \in \Omega$. Since Ω is a region there is a polygonal curve C joining w to any other point $z \in \Omega$. By the Fundamental Theorem for line integrals, we have

$$\int_C \vec{\nabla} u \cdot d\vec{r} = u(z) - u(w).$$

But $\vec{\nabla} u = \vec{0}$ and hence $u(z) - u(w) = 0$. In other words, we get $u(z) = u(w)$, for any $z \in \Omega$, where $u(w)$ is a constant. Therefore, we get

$$f(z) = u(z) + iv(z) = u(w) + ic \quad \forall z \in \Omega.$$

This implies that f is constant.

Problem 34

Assume $f = u + iv$ is analytic on a region Ω.

a) The derivative of f at z can be written as $f'(z) = u_x(z) + iv_x(z)$. Using the Cauchy-Riemann equations, we can rewrite the derivative $f'(z)$ in the following ways:

$$u_y = -u_x \Rightarrow f' = u_x - iu_y$$

and

$$u_x = v_y \Rightarrow f' = v_y + iv_x.$$

b) Using the formulas from part (a), we get

$$|f'|^2 = (u_x)^2 + (-u_y)^2 = u_x^2 + u_y^2$$

and

$$|f'|^2 = (v_y)^2 + (v_x)^2 = v_x^2 + v_y^2$$

as desired.