Problem 1

Here are two ways to solve this exercise:

① Consider z_1 and z_2 as points in the plane. So $z_1 = (1, 1)$ and $z_2 = (-1, -2)$. Then, we have to find the line segment passing through the points z_1 and z_2. The slope a is given by

$$a = \frac{\Delta y}{\Delta x} = \frac{-2-1}{-1-1} = \frac{3}{2}.$$

Therefore, any point $z = (x, y)$ on the line segment will by of the form $y = \frac{3}{2}x + b$, with $-1 \leq x \leq 1$. We will find b. We will use the fact that $z_1 = (1, 1)$ is on the line segment, so that

$$1 = \tfrac{3}{2}(1) + b \quad \Rightarrow \quad b = -\tfrac{1}{2}.$$

Therefore, $z = (x, y) = (x, \frac{3}{2}x - \frac{1}{2}) = x + (\frac{3}{2}x - \frac{1}{2})i$, for $-1 \leq x \leq 1$.

However, notice that this line segment starts at z_2 and terminates at z_1. To obtain the right parametrization, we use the reverse of the parametrization. Therefore, the interval is $[-1, 1]$ and so the reverse of the line segment is

$$\begin{aligned}\gamma(t) &= (1 + (-1) - t) + \left(\tfrac{3}{2}(1 + (-1) - t) - \tfrac{1}{2}\right)i \\ &= -t + (-\tfrac{3}{2}t - \tfrac{1}{2})i\end{aligned}$$

for $-1 \leq t \leq 1$.

② The second approach is a little more straightforward. The line segment from z_1 to z_2 is given by

$$\begin{aligned}\gamma(t) &= tz_2 + (1-t)z_1 \\ &= t(-1-2i) + (1-t)(1+i) \\ &= (-t + 1 - t) + (-2t + 1 - t)i \\ &= (1-2t) + (1-3t)i\end{aligned}$$

with $0 \leq t \leq 1$.

Problem 3

The equations of the parametrization of a circle centered at $3i = (0, 3)$ and of radius 1 is

$$x(t) = \cos(t) \quad \text{and} \quad y(t) = 3 + \sin(t) \quad (0 \leq t \leq 2\pi).$$

So, we get

$$\gamma(t) = \cos(t) + (3 + \sin(t))i = 3i + \cos(t) + i\sin(t) = 3i + e^{it}$$

with $0 \leq t \leq 2\pi$.

Problem 5

The parametrization of the unit circle is e^{it}, for $-\pi \leq t \leq \pi$. Therefore, the parametrization of the arc is

$$\gamma(t) = e^{it} \quad (-\tfrac{\pi}{4} \leq t \leq \tfrac{\pi}{4}).$$

Problem 6

We will compute the reverse curve of the curve in Exercise 5. Therefore, we get

$$\gamma(t) = e^{-it} \quad (-\tfrac{\pi}{4} \leq t \leq \tfrac{\pi}{4}).$$

Problem 9

A parametrization of the circle containing the arc in Figure 3.15 is

$$-3 + 2i + 5e^{it}$$

with $-\pi \leq t \leq \pi$. Then, a parametrization of the arc of the circle is

$$z(t) = -3 + 2i + 5e^{it} \quad (-\tfrac{\pi}{2} \leq t \leq 0).$$

Problem 11

The equation of a parabola is $y = x^2$. Set $x = t$, for $-1 \leq t \leq 1$. Therefore

$$\gamma(t) = x + iy = t + it^2$$

for $-1 \leq t \leq 1$.

Problem 12

The reverse is obtained by replacing t with $b + a - t = 0 + (-\frac{\pi}{2}) - t = -\frac{\pi}{2} - t$. Therefore

$$\begin{aligned} w(t) = z(-\tfrac{\pi}{2} - t) &= -3 + 2i + 5e^{i(-\frac{\pi}{2} - t)} \\ &= -3 + 2i + 5e^{-\frac{\pi}{2}i}e^{-it} \\ &= -3 + 2i - 5ie^{-it} \end{aligned}$$

for $-\frac{\pi}{2} \leq t \leq 0$.

Problem 14

The reverse is given by replacing t with $b + a - t = -t$. Therefore,

$$\gamma(t) = -t + it^2.$$

Problem 19

Using the product rule, we have

$$f'(t) = (t)'(e^{-it}) + t(e^{-it})' = e^{-it} - ite^{-it}.$$

Problem 22

Using the quotient rule:

$$\begin{aligned} f'(t) &= \frac{(2+i+t)'(-i-2t) - (2+i+t)(-i-2t)'}{(-i-2t)^2} \\ &= \frac{(1)(-i-2t) - (2+i+t)(-2)}{(i+2t)^2} \\ &= \frac{-i-2t+4+2i+2t}{(i+2t)^2} \\ &= \frac{4+i}{(i+2t)^2}. \end{aligned}$$

Problem 24

Notice that $F(z) = \text{Log}(z)$ is holomorphic on $\mathbb{C}\backslash(-\infty, 0]$. The curve $\gamma(t) = it$ in contained in $\mathbb{C}\backslash(-\infty, 0]$ whenever $t \neq 0$. Therefore $f(t) = F(\gamma(t))$, for $t \neq 0$ and hence

$$f'(t) = F'(\gamma(t))\gamma'(t) = \frac{1}{it}i = \frac{1}{t}.$$