Problem 2

We have

$$\begin{aligned}(2i+3+ix)(2i+3+ix) &= -4+6i-2x+6i+9+3ix-2x+3ix-x^2\\ &= 5-4x-x^2+i(12+6x).\end{aligned}$$

Therefore,

$$\int_{-1}^{1}(2i+3+ix)^2\,dx = \int_{-1}^{1}5-4x-x^2\,dx + i\int_{-1}^{1}12+6x\,dx = \tfrac{28}{3}+24i.$$

Problem 3

We have $\frac{d}{dx}(i\cos(ix)) = -i$

$$\int_{-1}^{0}\sin(ix)\,dx = i\cos(ix)|_{-1}^{0} = icos(i(0)) - icos(-i) = i - i\cosh(1) = -0.5430i.$$

Problem 5

Notice that

$$\frac{x+i}{x-i} = 1+\frac{2i}{x-i}$$

and the function $\operatorname{Log}(z-i)$ is analytic on $\mathbb{C}\backslash\{a+i\,:\,-\infty<a\leq 0\}$ with

$$\frac{d}{dz}\operatorname{Log}(z-i) = \frac{1}{z-i},$$

so $\frac{d}{dx}\operatorname{Log}(x-i) = \frac{1}{x-i}$. Therefore

$$\begin{aligned}\int_{-1}^{1}\frac{x+i}{x-i}\,dx = \int_{-1}^{1}1\,dx + 2i\int_{-1}^{1}\frac{1}{x-i}\,dt &= 2+2i\operatorname{Log}(x-i)|_{-1}^{1}\\ &= 2+2i(\operatorname{Log}(1-i)-\operatorname{Log}(-1-i)) \qquad = 2+2i(\tfrac{\pi}{2})\\ &= 2+i\pi.\end{aligned}$$

Problem 13

Write $z(t) = e^{it}$, $0\leq t\leq 2\pi$, so that

$$\begin{aligned}\int_{C_1(0)}(2z+i)\,dz = \int_{0}^{2\pi}(2e^{it}+i)ie^{it}\,dt = \int_{0}^{2\pi}2ie^{2it}-e^{it}\,dt &= e^{2it}+ie^{it}\Big|_{0}^{2\pi}\\ &= e^{4\pi i}+ie^{2\pi i}-e^{0}-ie^{0} = 0.\end{aligned}$$

Problem 14

Write $z(t) = i + e^{it}$, $0 \le t \le 2\pi$. Then

$$\begin{aligned}\int_{C_1(i)} z^2\, dz = \int_0^{2\pi} (i+e^{it})^2 ie^{it}\, dt &= \int_0^{2\pi} (-1+2ie^{it}+e^{2i})ie^{it}\, dt \\ &= \int_0^{2\pi} -ie^{it} - 2e^{2it} + ie^{3it}\, dt \\ &= -e^{it} + ie^{2it} + \tfrac{e^{3it}}{3}\Big|_0^{2\pi} \\ &= 0.\end{aligned}$$

Problem 17

We have $\gamma'(t) = ie^{it} - 2ie^{-it}$ and therefore, for $z = \gamma(t)$, we get

$$\gamma(t) + 2\overline{\gamma(t)} = e^{it} + 2e^{-it} + 2(e^{-it} + e^{it}) = 3e^{it} + 4e^{-it}$$

Hence, we have

$$\begin{aligned}\int_\gamma (z + 2\overline{z})\, dz &= \int_0^{2\pi} (3e^{it} + 4e^{-it})(ie^{it} - 2ie^{-it})\, dt \\ &= \int_0^{2\pi} 3ie^{2it} - 2i - 4ie^{-2it}\, dt \\ &= \tfrac{3}{2}e^{2it} - 2it + 2e^{-2it}\Big|_0^{2\pi} \\ &= -4\pi i.\end{aligned}$$

Problem 22

Let $\gamma(t) = e^{it}$, $0 \le t \le \pi$. Then

$$\int_\gamma z\, dz = \int_0^\pi e^{it} ie^{it}\, dt = \int_0^\pi ie^{2it}\, dt = \tfrac{e^{2it}}{2}\Big|_0^\pi = 0.$$

Problem 27

Notice that $\sqrt{z}$ is continuous on the quater circle $\gamma(t) = e^{it}$, $0 \le t \le \frac{\pi}{2}$. Therefore,

$$\int_\gamma \sqrt{z}\, dz = \int_0^{\pi/2} \sqrt{e^{it}} ie^{it}\, dt$$

Using the definition the principle value, we have $\sqrt{e^{it}} = e^{it/2}$, because $0 \le t \le \frac{\pi}{2}$ (in the range of the principle value of $\mathrm{Arg}(z)$). Therefore,

$$\int_\gamma \sqrt{z}\, dz = \int_0^{\pi/2} ie^{it/2}e^{it}\, dt = \int_0^{\pi/2} ie^{3it/2}\, dt = \frac{2}{3}e^{3it/2}\Big|_0^{\pi/2} = \tfrac{2}{3}(e^{3i\pi/4} - 1) = \tfrac{2}{3}(1 + i - 1) = \tfrac{2}{3}i.$$

Problem 31

We have
$$\gamma'(t) = 2 + it^{1/2}.$$
Therefore,
$$\ell(\gamma) = \int_1^2 |\gamma'(t)|\,dt = \int_1^2 \sqrt{4+t}\,dt = \int_5^6 \sqrt{u}\,du = \tfrac{2}{3}u^{3/2}\Big|_5^6 = \tfrac{2}{3}(6^{3/2} - 5^{3/2}) \approx 2.3444.$$