Problems: 1, 2, 3, 4, 7, 9, 11, 30.

Problem 1

The function $\cos(z)$ is analytic in the interior of and on $C_1(0)$. Since 0 is inside $C_1(0)$, we get

$$\int_{C_1(0)} \frac{\cos z}{z}\, dz = 2\pi i \cos(0) = 2\pi i$$

by Cauchy's Integral Formula.

Problem 2

The function $e^{z^2} \cos z$ is analytic in the interior of and on $C_3(0)$. Since i is inside the curve $C_3(0)$, we get

$$\int_{C_3(0)} \frac{e^{z^2} \cos z}{z - i}\, dz = 2\pi e^{i^2} \cos(i) = 2\pi e^{-1}\Big(\frac{e^{-1} + e^1}{2}\Big) = \pi(e^{-2} + 1)$$

by Cauchy's Integral Formula.

Problem 3

Notice that $z^2 - 5z + 4 = (z-4)(z-1)$. The number $z = 4$ is outside of $C_2(1)$, but $z = 1$ is inside $C_2(1)$. Therefore, by Cauchy's Integral Formula,

$$\frac{1}{2\pi i}\int_{C_2(1)} \frac{1}{z^2 - 5z + 4}\, dz = \frac{1}{2\pi i}\int_{C_2(1)} \frac{1}{(z-4)(z-1)}\, dz = \frac{1}{2\pi i}\int_{C_2(1)} \frac{1/(z-4)}{z-1}\, dz = \frac{1}{1-4} = -\frac{1}{3}.$$

Problem 5

The function $\operatorname{Log}(z)$ is analytic on $\mathbb{C}\backslash(-\infty, 0]$ and therefore it is analytic on the inside of and on $C_{\frac{1}{2}}(i)$. From Cauchy's Integral Formula,

$$\int_{C_{\frac{1}{2}}(i)} \frac{\operatorname{Log} z}{-z + i}\, dz = -\int_{C_{\frac{1}{2}}(i)} \frac{\operatorname{Log} z}{z - i}\, dz = -2\pi i \operatorname{Log}(i) = -\pi^2 i.$$

Problem 7

The Jordan curve $\gamma = [z_1, z_2, z_3, z_1]$ is a triangle. From the picture below, we can see that the point $z = 1$ lies in the inside of γ.

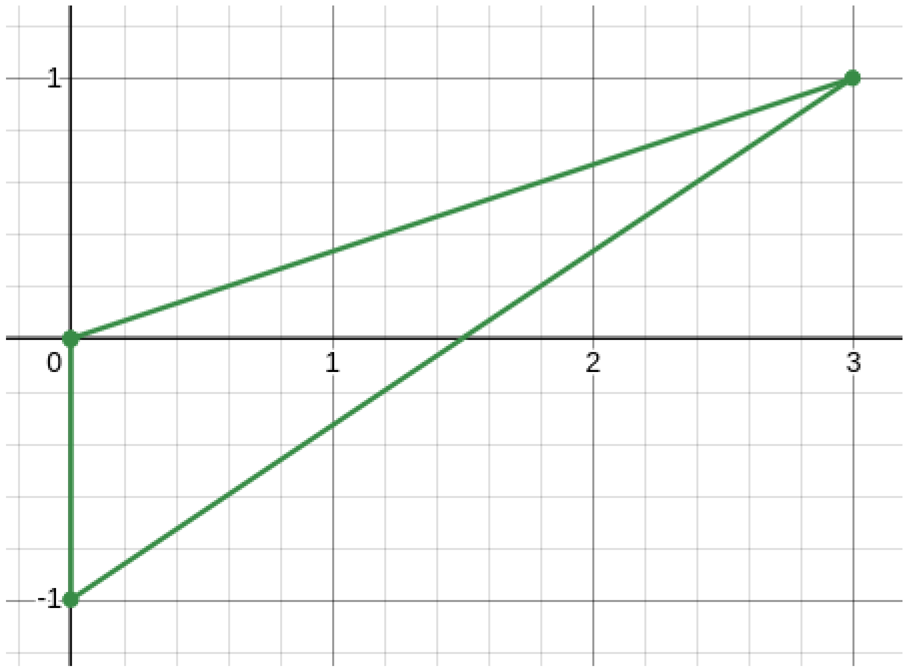

Figure 1: Triangle $[z_1, z_2, z_3, z_1]$

The function z^{19} is analytic on the inside of and on the triangle γ and from Generalized Cauchy's Integral Formula

$$\frac{18!}{2\pi i}\int_{[z_1,z_2,z_3,z_1]} \frac{z^{19}}{(z-1)^{19}}\,dz = f^{(18)}(1).$$

We have $f^{(18)}(z) = (19)(18)\cdots(2)z = (19!)z$ and hence

$$\int_{[z_1,z_2,z_3,z_1]} \frac{z^{19}}{(z-1)^{19}}\,dz = \frac{2\pi i}{18!}(19!)(1) = 38\pi i.$$

Problem 9

The function $\sin z$ is analytic on the interior of and on γ. Using generalized Cauchy's Integral Formula, we get

$$\frac{2!}{2\pi i}\int_\gamma \frac{\sin z}{(z-\pi)^3}\,dz = \frac{d^2}{dz^2}(\sin z)\bigg|_{z=\pi} = -\sin(\pi) = 0$$

and therefore

$$\int_\gamma \frac{\sin z}{(z-\pi)^3}\,dz = 0.$$

Problem 11

As it is, we can't apply Cauchy's Integral Formula because the integrand is not in the right form[1]. Instead, we will try to modify the path γ to isolate each singularities.

Consider the construction:

[1]We can't use partial fraction decomposition here.

- Let C_1 and C_2 be two circles centered at 0 and π respectively with radii small enough so that C_1 and C_2 do not intersect and $C_1 \cup C_2 \subset \Omega^-$, where Ω^- is the interior of the curve γ.
- Let $[z_i, z_{i+1}]$ $(i = 1, 3, 5, 7)$ be line segments with starting point on γ and ending point on $C_1 \cup C_2$ as illustrated in the picture below.
- Let w_1, w_2, w_3, w_4 be the vertices of γ such that $\gamma = [w_1, w_2, w_3, w_4, w_1]$.
- Let ζ_1, ζ_2, ζ_3, and ζ_4 be points on the circles C_1 and C_2 as illustrated on the picture below. We use the notation (A, B, C) for the arc starting at A, ending at C and passing through B.

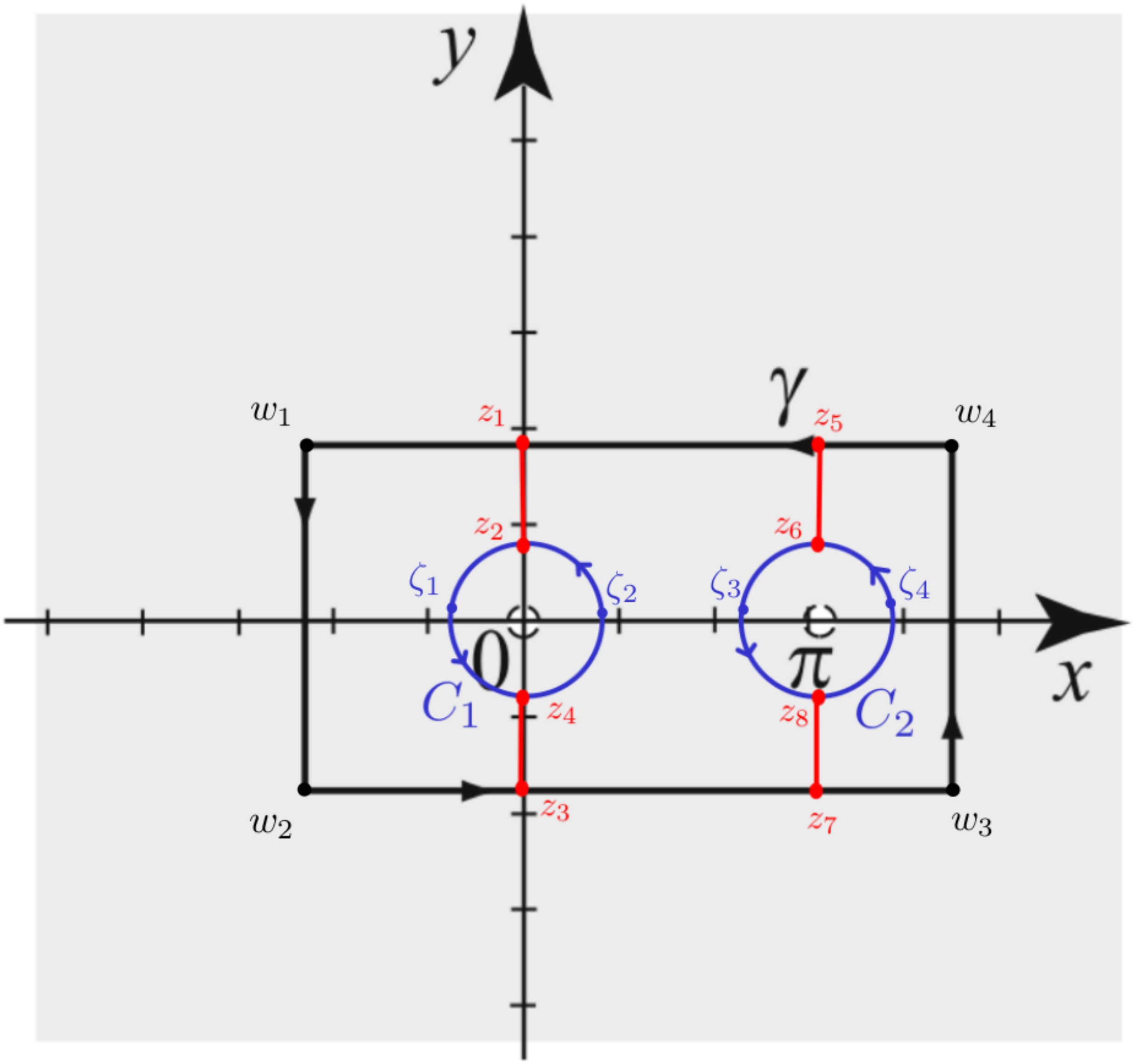

Figure 2: Reconfiguration of the path γ

Let $f(z) = \frac{e^z \sin z}{z^2(z-\pi)}$. Firstly, f is analytic on the interior of $\Gamma_1 := [z_2, z_1, w_1, w_2, z_3, z_4] \cup (z_4, \zeta_1, z_2)$. By Cauchy's Theorem, we get

$$\int_{\Gamma_1} f(z)\, dz = 0. \tag{1}$$

Secondly, f is analytic on the interior of $\Gamma_2 := [z_6, z_5, z_1, z_2] \cup (z_2, \zeta_2, z_4) \cup [z_4, z_3, z_7, z_8] \cup (z_8, \zeta_3, z_6)$. By Cauchy's Theorem again, we get

$$\int_{\Gamma_2} f(z)\,dz = 0. \tag{2}$$

Thirdly, f is analytic on the interior of $\Gamma_3 := [z_8, z_7, w_3, w_4, z_5, z_6] \cup (z_6, \zeta_4, z_8)$. Then, by Cauchy's Theorem, we get

$$\int_{\Gamma_3} f(z)\,dz = 0. \tag{3}$$

Adding up (1), (2), and (3), we get

$$\int_{\Gamma_1} f(z)\,dz + \int_{\Gamma_2} f(z)\,dz + \int_{\Gamma_3} f(z)\,dz = 0.$$

After canceling out the path traversed twice, we obtain

$$\int_{\gamma} f(z)\,dz - \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$

which gives

$$\int_{\gamma} \frac{e^z \sin z}{z^2(z-\pi)}\,dz = \int_{C_1} \frac{e^z \sin z}{z^2(z-\pi)}\,dz + \int_{C_2} \frac{e^z \sin z}{z^2(z-\pi)}\,dz.$$

① On the interior of C_1, the function $z \mapsto \frac{e^z \sin z}{z-\pi}$ is analytic on the interior of and on C_1. By the Generalized Cauchy's Integral formula, we get

$$\frac{1}{2\pi i}\int_{C_1} \frac{\frac{e^z \sin z}{z-\pi}}{z^2}\,dz = \frac{d}{dz}\left(\frac{e^z \sin z}{z-\pi}\right)\bigg|_{z=0} = \left(\frac{e^z \sin z + e^z \cos z}{z-\pi} - \frac{e^z \sin z}{(z-\pi)^2}\right)\bigg|_{z=0} = \frac{1}{1-\pi}$$

and hence

$$\int_{C_1} \frac{e^z \sin z}{z^2(z-\pi)}\,dz = \frac{2\pi i}{1-\pi}.$$

② On the interior of and on C_2, the function $z \mapsto \frac{e^z \sin z}{z^2}$ is analytic. By Cauchy's Integral Formula, we get

$$\frac{1}{2\pi i}\int_{C_2} \frac{\frac{e^z \sin z}{z^2}}{z-\pi}\,dz = \frac{e^\pi \sin \pi}{\pi^2} = 0.$$

Hence,

$$\int_{C_2} \frac{e^z \sin z}{z^2(z-\pi)}\,dz = 0.$$

Therefore, we obtain

$$\int_{\gamma} \frac{e^z \sin z}{z^2(z-\pi)}\,dz = \frac{2\pi i}{1-\pi}.$$

Problem 30

Assume that f is analytic on the interior and on $C_1(0)$ and assume $|z| < 1$. Then $|1/z| > 1$ and therefore the number $1/z$ is outside of $C_1(0)$. This means the function $g(w) = \frac{f(w)}{w-1/z}$ is analytic on the disk $B_r(0)$ with $r = 1/|z| > 1$. Hence, g is analytic on the interior of and on $C_1(0)$. By Cauchy's Theorem, we get

$$\int_{C_1(0)} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \int_{C_1(0)} g(\zeta)\, d\zeta = 0. \qquad \square$$