Problems: 9, 11, 13, 14, 16, 18, 20.

Problem 9

(a) Assume that $|z| \leq 1$. Then we have

$$\lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} \Big(\frac{1}{n}\Big)\Big(\frac{z+1/n}{z/n^2+2}\Big) = (0)(z/2) = 0.$$

Therefore $f_n \to 0$ on $\overline{B_1(0)}$.

(b) If $|z| \leq 1$, notice that

$$|nz+1| \leq n|z| + 1 \leq n+1$$

and

$$|z+2n^2| \geq 2n^2 - |z| \geq 2n^2 - 1.$$

Therefore, for any $|z| \leq 1$, we get

$$|f_n(z)| \leq \frac{n+1}{2n^2-1} \quad \Rightarrow \quad \lim_{n\to\infty} \max_{|z|\leq 1} |f_n(z)| \leq \lim_{n\to\infty} \frac{n+1}{2n^2-1} = 0.$$

Hence $f_n \rightrightarrows 0$ on $\overline{B_1(0)}$.

(c) The uniform convergence did not fail in this case.

Problem 11

(a) Notice that if $z = i$, then

$$f_n(i) = \frac{e^{-n} - e^{n}}{2in^2}.$$

Therefore,

$$\lim_{n\to\infty} |f_n(i)| = \lim_{n\to\infty} \frac{e^{n} - e^{-n}}{2n^2} = \infty.$$

So the limit does not exist. In fact, for any $z = x+iy$ with $y \neq 0$ and $|z| \leq 1$, we have

$$\lim_{n\to\infty} |f_n(z)| = \infty.$$

Indeed, $f_n(z) = \frac{e^{inx-ny} - e^{-inx+ny}}{2in^2}$ and therefore

$$|f_n(z)| = \frac{|e^{inx-ny} - e^{-inx+ny}|}{2n^2} \geq \frac{||e^{-inx+ny}| - |e^{inx-ny}||}{2n^2} = \frac{|e^{ny} - e^{-ny}|}{2n^2} = \frac{|\sinh(ny)|}{n^2}$$

Because $y \neq 0$, we have $\lim_{n\to\infty} |\sinh(ny)| = \infty$ which proves the claim. Therefore, the functions f_n do not converge pointwise on $\overline{B_1(0)}$.

(b) Since f_n do not converge pointwise on $\overline{B_1(0)}$, they can't converge uniformly.

(c) Let $E = [-1, 1]$. Then we have $nz \in \mathbb{R}$ and so that $|\sin(nz)| \leq 1$. Therefore for $z \in [-1, 1]$

$$|f_n(z)| \leq \frac{1}{n^2} \quad \Rightarrow \quad \lim_{n\to\infty} \max_{-1\leq x\leq 1} |f_n(z)| \leq \lim_{n\to\infty} \frac{1}{n^2} = 0.$$

Hence $f_n \rightrightarrows 0$ on $[-1, 1]$.

Problem 13

We have $u_n(z) = \frac{z^n}{n(n+1)}$ for $|z| \leq 1$. Notice that

$$|u_n(z)| \leq \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

The series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) = 1.$$

By the Weierstrass M-test, the series $\sum_{n=1}^{\infty} \frac{z^n}{n(n+1)}$ converges uniformly on $\overline{B_1(0)}$.

Problem 14

We have $u_n(z) = \frac{z^n}{n(n+1)}$ for $|z| \leq 1/3$. Notice that $|z| \leq 1/3 \iff 3|z| \leq 1$. Therefore

$$|u_n(z)| \leq \frac{1}{n(n+1)}.$$

Now we have

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 < \infty.$$

By the Weierstrass M-test, the series $\sum_{n=1}^{\infty} \frac{(3z)^n}{n(n+1)}$ converges uniformly on $\overline{B_{1/3}(0)}$.

Problem 16

We have $u_n(z) = \left(\frac{z^2-1}{4}\right)^n$ for $|z| \leq 1$. Notice that

$$|u_n(z)| = \left(\frac{|z^2-1|}{4}\right)^n \leq \left(\frac{|z|^2+1}{4}\right)^n \leq \left(\frac{1+1}{4}\right)^n = \left(\frac{1}{2}\right)^n.$$

The series $\sum_{n=0}^{\infty} \frac{1}{2^n}$ is convergent (a geometric series with $z = 1/2 < 1$). Therefore, by the Weierstrass M-test, the series $\sum_{n=0}^{\infty} \left(\frac{z^2-1}{4}\right)^n$ converges uniformly on $\overline{B_1(0)}$.

Problem 18

We have $u_n(z) = \frac{1}{(5-z)^n}$ for $|z| \leq \frac{7}{2}$. First, notice that for $|z| \leq 7/2$

$$|5 - z|^n \geq (5 - |z|)^n \geq (5 - 3.5)^n = \left(\frac{3}{2}\right)^n.$$

Hence, we get

$$|u_n(z)| = \frac{1}{|5-z|^n} \le \left(\frac{2}{3}\right)^n.$$

The series $\sum_{n=0}^{\infty}(\frac{2}{3})^n$ is convergent (a geometric series with $z = 2/3 < 1$). Therefore, by the Weierstrass M-test, the series $\sum_{n=0}^{\infty} \frac{1}{(5-z)^n}$ converges uniformly on $|z| \le 7/2$.

Problem 20

Notice that $|z-1| \le |z| + 1 \le 3$ when $|z| \le 2$. Therefore, for $|z| \le 2$, we have

$$\left|\frac{(z-1)^n}{4^n}\right| \le \frac{3^n}{4^n} = \left(\frac{3}{4}\right)^n.$$

The series $\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n$ is convergent (a geometric series with $z = 3/4 < 1$). Therefore, by the Weierstrass M-test, the series $\sum_{n=0}^{\infty} \frac{(z-1)^n}{4^n}$ converges uniformly on $\overline{B_2(0)}$.