Problems: 1, 2, 3, 5, 6, 13, 19, 23, 25, 26.

Problem 1

We write

$$\frac{1}{1+z} = \frac{1}{z(1/z+1)} = \frac{1}{z}\Big(\frac{1}{1-(-\frac{1}{z})}\Big).$$

Since $|z| > 1$, we have $|1/z| < 1$, so

$$\frac{1}{1-(-\frac{1}{z})} = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^n}$$

and hence

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}}$$

when $|z| > 1$.

Problem 2

We have

$$\frac{1}{2+iz} = \frac{1}{iz(\frac{2}{iz}+1)} = \frac{1}{iz}\Big(\frac{1}{1-(-\frac{2}{iz})}\Big).$$

since $|z| > 2$, we have $|iz| > 2$, so that $\frac{2}{|iz|} < 1$ and therefore

$$\frac{1}{1-(-\frac{2}{iz})} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{i^n z^n} = \sum_{n=0}^{\infty} \frac{i^{2n} 2^n}{i^n z^n} = \sum_{n=0}^{\infty} \frac{i^n 2^n}{z^n}.$$

Hence

$$\frac{1}{2+iz} = \sum_{n=0}^{\infty} \Big(\frac{-i^{n+1} 2^n}{z^{n+1}}\Big).$$

Problem 3

Notice that

$$3 + 2iz = 2i\Big(-\frac{3i}{2} + z\Big) = 2i\Big(-\frac{5i}{2} + z + i\Big) = 2i(z+i)\Big(1 - \frac{5i/2}{z+i}\Big).$$

Since $|z+i| > 5/2$, then

$$\frac{|5i/2|}{|z+i|} = \frac{5/2}{|z+i|} < 1.$$

Therefore, we get

$$\frac{1}{3+2iz} = \frac{1}{2i(z+i)}\left(\frac{1}{1-\frac{5i/2}{z+i}}\right) = \frac{1}{2i(z+i)}\sum_{n=0}^{\infty}\frac{(5i/2)^n}{(z+i)^n}$$
$$= \sum_{n=0}^{\infty}\frac{-5^n i^{n+1}/2^{n+1}}{(z+i)^{n+1}}$$

Problem 5

Since $|z| > 1$ we have $|z^2| > 1$, so that $|1/z^2| < 1$ and therefore

$$\frac{1}{1+z^2} = \frac{1}{z^2}\left(\frac{1}{1-(-1/z^2)}\right) = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n+2}}.$$

Problem 6

Using partial fraction decomposition, we write

$$\frac{1}{1-z^2} = \frac{1}{(1-z)(1+z)} = \frac{1/2}{1-z} + \frac{1/2}{1+z}.$$

We have

$$\frac{1}{1-z} = \frac{1}{-1-(z-2)} = \frac{1}{z-2}\left(\frac{1}{-1-\frac{1}{z-2}}\right) = \frac{-1}{z-2}\left(\frac{1}{1-(-\frac{1}{z-2})}\right)$$

and since $|z-2| > 1$, then

$$\frac{1}{1-z} = \frac{-1}{z-2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(z-2)^n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(z-2)^{n+1}}.$$

For the second term, we have

$$\frac{1}{1+z} = \frac{1}{3+z-2} = \frac{1}{3}\left(\frac{1}{1+\frac{z-2}{3}}\right).$$

Since $|z-2| < 3$, we have $|z-3|/3 < 1$ and then

$$\frac{1}{1+z} = \frac{1}{3}\sum_{n=0}^{\infty}\frac{(z-2)^n}{3^n} = \sum_{n=0}^{\infty}\frac{(z-2)^n}{3^{n+1}}.$$

Hence, for $1 < |z-2| < 3$, we get

$$\frac{1}{1-z^2} = \sum_{n=0}^{\infty}\frac{(z-2)^n}{23^{n+1}} + \sum_{n=0}^{\infty}\frac{(-1)^n/2}{(z-2)^{n+1}}.$$

Problem 13

We write

$$\frac{z}{(z+2)(z+3)} = \frac{-2}{z+2} + \frac{3}{z+3}.$$

For the first part, since $|z| > 2$, we have $|2/z| < 1$. Therefore

$$\frac{1}{z+2} = \frac{1}{z}\Big(\frac{1}{1+\frac{2}{z}}\Big) = \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-2)^n}{z^n} = \sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{z^{n+1}}.$$

For the second part, since $|z| < 3$, we have $|z/3| < 1$. Therefore

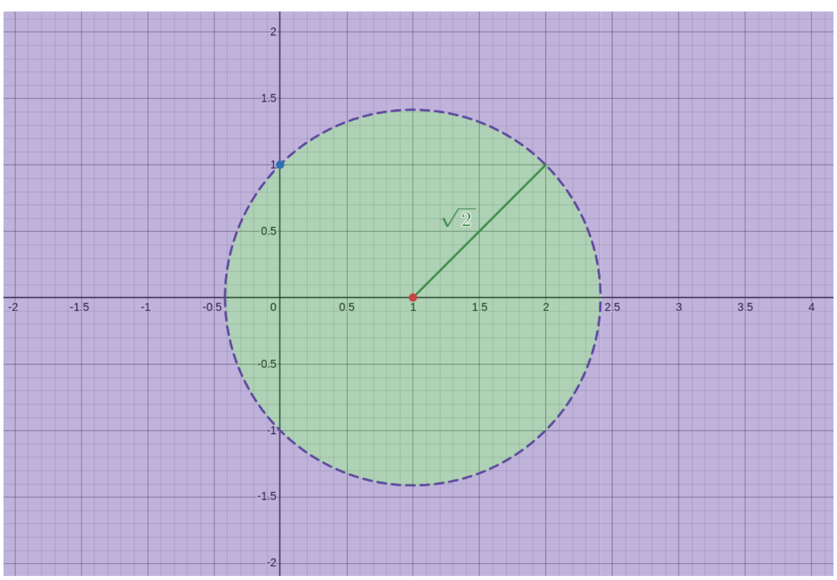
$$\frac{1}{z+3} = \frac{1}{3}\Big(\frac{1}{1-(-z/3)}\Big) = \sum_{n=0}^{\infty}\frac{(-1)^n z^n}{3^{n+1}}.$$

Replacing in the partial fraction decomposition of the function and for $2 < |z| < 3$, we get

$$\frac{z}{(z+2)(z+3)} = \sum_{n=0}^{\infty}\frac{(-1)^n z^n}{3^{n+1}} + \sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{z^{n+1}}.$$

Problem 19

There are two regions to deal with: $B_{\sqrt{2}}(1)$ and $A_{\sqrt{2},\infty}(1)$. There are illustrated in the figure below.



$B_{\sqrt{2}}(1)$ in green and $A_{\sqrt{2},\infty}(1)$ in purple

On $\mathbf{B_{\sqrt{2}}(1)}$. We write

$$\frac{1}{z+i} = \frac{1}{1+i+z-1} = \frac{1}{1+i}\left(\frac{1}{1+\frac{z-1}{1+i}}\right).$$

We have $|z-1| < \sqrt{2}$ and so $|(z-1)/(1+i)| < 1$. Hence

$$\frac{1}{z+i} = \frac{1}{1+i}\left(\frac{1}{1+\frac{z-1}{1+i}}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(1+i)^{n+1}}(z-1)^n.$$

On $\mathbf{A_{\sqrt{2},\infty}(1)}$. We write

$$\frac{1}{z+i} = \frac{1}{1+i+z-1} = \frac{1}{z-1}\left(\frac{1}{1+\frac{1+i}{z-1}}\right).$$

We have $|z-1| > \sqrt{2}$ so that $|1+i|/|z-1| < 1$. Hence

$$\frac{1}{z+i} = \frac{1}{z-1}\left(\frac{1}{1+\frac{1+i}{z-1}}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n(1-i)^n}{(z-1)^{n+1}}.$$

Problem 23

(a) Since $|z| > 1$, we have $1/|-z| < 1$. Therefore, we get

$$\frac{1}{1+z} = \frac{1}{z}\left(\frac{1}{1+\frac{1}{z}}\right) = \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}} = \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{z^m}.$$

(b) The above series converges uniformly on each closed disk of the annular region $A_{1,\infty}(0)$. We can therefore take the derivative of the series term-by-term:

$$\frac{d}{dz}\left(\frac{1}{1+z}\right) = \sum_{m=1}^{\infty}(-1)^{m-1}\frac{d}{dz}\left(\frac{1}{z^m}\right)$$

and therefore

$$\frac{-1}{(1+z)^2} = \sum_{m=1}^{\infty}(-1)^{m-1}\left(\frac{-m}{z^{m+1}}\right) \quad \Rightarrow \quad \frac{1}{(1+z)^2} = \sum_{m=1}^{\infty}\frac{m(-1)^{m-1}}{z^{m+1}}$$

$$\Rightarrow \quad \frac{1}{(1+z)^2} = \sum_{k=2}^{\infty}\frac{(k-1)(-1)^{k-2}}{z^k}.$$

(c) We multiply by z the series obtained in (b):

$$\frac{z}{(1+z)^2} = \sum_{k=2}^{\infty}\frac{(k-1)(-1)^{k-2}}{z^{k-1}} = \sum_{m=1}^{\infty}\frac{m(-1)^{m-1}}{z^m}.$$

(d) We first find the expansion of $1/(1+z)^3$. We take the derivative of the series expansion in (b). We get

$$\frac{-2}{(1+z)^3} = \sum_{k=2}^{\infty}(k-1)(-1)^{k-2}\frac{d}{dz}\left(\frac{1}{z^k}\right) = \sum_{k=2}^{\infty}(k-1)(-1)^{k-2}\frac{-k}{z^{k+1}}.$$

Therefore

$$\frac{1}{(1+z)^3} = \sum_{k=2}^{\infty}\frac{k(k-1)(-1)^{k-2}}{z^{k+1}} \quad\Rightarrow\quad \frac{z^2}{(1+z)^3} = \sum_{k=2}^{\infty}\frac{k(k-1)(-1)^{k-2}}{z^{k-1}}$$
$$\Rightarrow\quad \frac{z^2}{(1+z)^3} = \sum_{k=2}^{\infty}\frac{k(k-1)(-1)^{k}}{z^{k-1}}.$$

Problem 25

Replacing z by $1/z$ in the power series for $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i} = \sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!}$, when $z \neq 0$, we get

$$\begin{aligned}\int_{C_1(0)} \sin(1/z)\,dz = \int_{C_1(0)} \sum_{n=0}^{\infty}\frac{(1/z)^{2n+1}}{(2n+1)!}\,dz &= \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\int_{C_1(0)}\frac{1}{z^{2n+1}}\,dz\\ &= \underbrace{\int_{C_1(0)}\frac{1}{z}\,dz}_{n=0} + \underbrace{\frac{1}{3!}\int_{C_1(0)}\frac{1}{z^3}\,dz}_{n=1} + \cdots\\ &= 2\pi i + 0 + 0 + \cdots\\ &= 2\pi i.\end{aligned}$$

Problem 26

Replacing z by $1/z^2$ in the power series of $\cos(z)$, $z \neq 0$, we get

$$\frac{\cos(\frac{1}{z^2})}{z} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{(1/z^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}\frac{1}{(2n)!z^{4n+1}}.$$

Since the convergence of the series is uniform on $C_1(0)$, we obtain

$$\begin{aligned}\int_{C_1(0)}\frac{\cos(1/z^2)}{z}\,dz = \sum_{n=0}^{\infty}\frac{1}{(2n)!}\int_{C_1(0)}\frac{1}{z^{4n+1}}\,dz &= \underbrace{\int_{C_1(0)}\frac{1}{z}\,dz}_{n=0} + \underbrace{\frac{1}{2!}\int_{C_1(0)}\frac{1}{z^5}\,dz}_{n=1} + \underbrace{\frac{1}{4!}\int_{C_1(0)}\frac{1}{z^9}\,dz}_{n=2} + \cdots\\ &= 2\pi i + 0 + 0 + \cdots\\ &= 2\pi i.\end{aligned}$$