Problems: 2, 4, 5.

## Problem 2

Let $z = e^{i\theta}$, so that

$$\int_0^{2\pi} \frac{1}{5+3\cos\theta}\,d\theta = -i\int_{C_1(0)} \frac{1}{5+3\frac{z^2+1}{2z}}\,\frac{dz}{z} = -i\int_{C_1(0)} \frac{2}{3z^2+10z+3}\,dz.$$

The function $f(z) = \frac{2}{3z^2+10z+3}$ has simple poles at $z=-3$ and $z=-1/3$. Only $z=-1/3$ is inside $C_1(0)$. Using Cauchy's Residue Theorem, we get

$$\begin{aligned}
-i\int_{C_1(0)} \frac{2}{3z^2+10z+3}\,dz &= -i(2\pi i)\,\mathrm{Res}(f(z), -1/3) \\
&= 2\pi \lim_{z\to -1/3} \frac{2(z+1/3)}{3(z+1/3)(z+3)} \\
&= \frac{4\pi}{3} \lim_{z\to -1/3} \frac{1}{z+3} \\
&= \frac{4\pi}{3}\left(\frac{3}{8}\right) \\
&= \frac{\pi}{2}.
\end{aligned}$$

## Problem 4

Let $z = e^{i\theta}$, so that

$$\int_0^{2\pi} \frac{1}{\sin^2\theta + 2\cos^2\theta}\,d\theta = -i\int_{C_1(0)} \frac{1}{\left(\frac{z^2-1}{2iz}\right)^2 + 2\left(\frac{z^2+1}{2z}\right)^2}\,\frac{dz}{z} = -i\int_{C_1(0)} \frac{4z}{z^4+6z^2+1}\,dz.$$

The poles of the function $\frac{4z}{z^4+6z^2+1}$ are

$$z_\pm = \pm\Big(i\sqrt{3-2\sqrt{2}}\Big) \quad \text{and} \quad w_\pm = \pm\Big(i\sqrt{3+2\sqrt{2}}\Big).$$

Only $z_+$ and $z_-$ are inside $C_1(0)$. Therefore, by Cauchy's Residue Theorem, we obtain

$$\int_0^{2\pi} \frac{1}{\sin^2\theta + 2\cos^2\theta}\,d\theta = -i(2\pi i)\Big(\mathrm{Res}(f(z), z_+) + \mathrm{Res}(f(z), z_-)\Big)$$

The poles are simple and from calculations similar to the ones in the previous problem, we get

$$\mathrm{Res}(f(z), z_+) = \frac{2}{(z_+ - w_+)(z_+ + w_+)} = \frac{2}{z_+^2 - w_+^2} = \frac{1}{2\sqrt{2}}$$

and

$$\operatorname{Res}(f(z), z_-) = -\frac{2}{(z_+ + w_+)(w_+ - z_+)} = \frac{2}{z_+^2 - w_+^2} = \frac{1}{2\sqrt{2}}.$$

Hence,

$$\int_0^{2\pi} \frac{1}{\sin^2\theta + 2\cos^2\theta}\, d\theta = 2\pi\Big(\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\Big) = \pi\sqrt{2}.$$

## Problem 5

Let $z = e^{i\theta}$. However, $\cos(2\theta)$ is not of the form $\cos\theta$. We have to change it in the following way:

$$\cos(2\theta) = \frac{e^{2i\theta} + e^{-2i\theta}}{2} = \frac{z^2 + z^{-2}}{2} = \frac{z^4 + 1}{2z^2}.$$

We can then substitute that into the integral:

$$\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta}\, d\theta = -i\int_{C_1(0)} \frac{\frac{z^4+1}{2z^2}}{5 + 4\frac{z^2+1}{2z}}\,\frac{dz}{z} = -i\int_{C_1(0)} \frac{z^4 + 1}{z^2(4z^2 + 10z + 4)}\, dz.$$

The function $f(z) = \frac{z^4+1}{z^2(4z^2+10z+4)}$ has poles inside $C_1(0)$ at $z = 0$ and $z = -1/2$. Therefore, by Cauchy's Residue Theorem, we get

$$-i\int_{C_1(0)} \frac{z^4 + 1}{z^2(4z^2 + 10z + 4)}\, dz = -i(2\pi i)\Big(\operatorname{Res}(f(z), 0) + \operatorname{Res}(f(z), -1/2)\Big).$$

Since $z = 0$ is a pole of order $m = 2$, we have

$$\begin{aligned}\operatorname{Res}(f(z), 0) &= \lim_{z\to 0} \frac{d}{dz}\Big(\frac{z^2(z^4+1)}{z^2(4z^2 + 10z + 4)}\Big)\\ &= \lim_{z\to 0} \frac{d}{dz}\Big(\frac{z^4+1}{4z^2 + 10z + 4}\Big)\\ &= \lim_{z\to 0} \frac{-5 - 4z + 8z^3 + 15z^4 + 4z^5}{2(2 + 5z + 2z^2)^2}\\ &= \frac{-5}{8}\end{aligned}$$

and since $z = -1/2$ is a pole of order $m = 1$, we have

$$\operatorname{Res}(f(z), -1/2) = \lim_{z\to 0} \frac{(z + 1/2)(z^4 + 1)}{4z^2(z + 1/2)(z + 2)} = \lim_{z\to -1/2} \frac{z^4 + 1}{4z^2(z + 2)} = \frac{17}{24}.$$

Hence

$$\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta}\, d\theta = 2\pi\Big(\frac{-15 + 17}{24}\Big) = \frac{\pi}{6}.$$