Problems: 1, 2, 5, 7.

Problem 1

From the formula

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2 + a^2} = \frac{\pi \coth(a\pi)}{a}$$

with $a = 3$, we get

$$\sum_{k=-\infty}^{\infty} \frac{1}{k^2 + 9} = \frac{\pi \coth(3\pi)}{3}.$$

Problem 2

Let $f(z) = \frac{1}{(z^2+1)^2}$. Then the poles of f are at $-i$ and i only. There are of order 2. Hence, we get

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k^2 + 1)^2} = -\pi \operatorname{Res}(f(z)\cot(\pi z), -i) - \pi \operatorname{Res}(f(z)\cot(\pi z), i).$$

Since $-i$ is a pole of order 2, we have

$$\begin{aligned}
\operatorname{Res}(f(z)\cot(\pi z), -i) &= \lim_{z\to -i} \frac{d}{dz}\Big((z+i)^2 \frac{\cot(\pi z)}{(z^2+1)^2} \Big) \\
&= \lim_{z\to -i} \frac{d}{dz}\Big(\frac{\cot(\pi z)}{(z-i)^2} \Big) \\
&= \lim_{z\to -i} \frac{-\pi \csc^2(\pi z)(z-i)^2 - 2(z-i)\cot(\pi z)}{(z-i)^4} \\
&= \lim_{z\to -i} \frac{-\pi \csc^2(\pi z)(z-i) - 2\cot(\pi z)}{(z-i)^3} \\
&= \frac{-\pi \csc^2(-\pi i)(-2i) - 2\cot(-\pi i)}{-8i} \\
&= \frac{\pi \operatorname{csch}^2(\pi)(2i) + 2i\coth(\pi)}{-8i} \\
&= -\frac{\pi \operatorname{csch}^2(\pi) + \coth(\pi)}{4}
\end{aligned}$$

Similarly, we have

$$\operatorname{Res}(f(z)\cot(\pi z), i) = -\frac{\pi \operatorname{csch}^2(\pi) + \coth(\pi)}{4}.$$

Hence

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k^2+1)^2} = -\pi\Big(-\frac{\pi\,\mathrm{csch}^2(\pi)+\coth(\pi)}{4} - \frac{\pi\,\mathrm{csch}^2(\pi)+\coth(\pi)}{4}\Big)$$
$$= \frac{\pi^2\,\mathrm{csch}^2(\pi)}{2} + \frac{\pi\coth(\pi)}{2}.$$

Problem 5

We have $f(z) = \frac{1}{4z^2-1}$ has poles at $\frac{1}{2}$ and $-\frac{1}{2}$. There are simple poles.

From the formula in the lecture notes, we get

$$\sum_{k=-\infty}^{\infty} \frac{1}{4k^2-1} = -\pi\,\mathrm{Res}\Big(f(z)\cot(\pi z), \frac{1}{2}\Big) - \pi\,\mathrm{Res}\Big(f(z)\cot(\pi z), -\frac{1}{2}\Big).$$

We have

$$\mathrm{Res}\Big(f(z)\cot(\pi z), \frac{1}{2}\Big) = \lim_{z\to 1/2}\Big(z-\frac{1}{2}\Big)\frac{\cot(\pi z)}{4(z-1/2)(z+1/2)} = \cot(\pi/2) = 0.$$

Similarly, we get

$$\mathrm{Res}\Big(f(z)\cot(\pi z), -\frac{1}{2}\Big) = \lim_{z\to -1/2}\Big(z+\frac{1}{2}\Big)\frac{\cot(\pi z)}{4(z+1/2)(z-1/2)} = \cot(-\pi/2) = 0.$$

Hence,

$$\sum_{k=-\infty}^{\infty} \frac{1}{4k^2-1} = 0.$$

Notice that

$$\sum_{k=-\infty}^{\infty} \frac{1}{4k^2-1} = -1 + 2\sum_{k=1}^{\infty} \frac{1}{4k^2-1}$$

and therefore

$$\sum_{k=1}^{\infty} \frac{1}{4k^2-1} = \frac{1}{2}.$$

Neat hey!

Problem 7

Let $f(z) = \frac{1}{(4z^2-1)^2}$. Then $f(z)$ has poles at $z = 1/2$ and $z = -1/2$ of order 2.

From the formula in the lecture notes, we get

$$\sum_{k=-\infty}^{\infty} \frac{1}{(4k^2-1)^2} = -\pi\,\mathrm{Res}(f(z)\cot(\pi z), 1/2) - \pi\,\mathrm{Res}(f(z)\cot(\pi z), -1/2).$$

We can compute that

$$
\begin{aligned}
\operatorname{Res}(f(z)\cot(\pi z), 1/2) &= \lim_{z\to 1/2} \frac{d}{dz}\Big(\frac{(z-1/2)^2\cot(\pi z)}{16(z-1/2)^2(z+1/2)^2}\Big) \\
&= \lim_{z\to 1/2} \frac{d}{dz}\Big(\frac{\cot(\pi z)}{16(z+1/2)^2}\Big) \\
&= \lim_{z\to 1/2} \frac{-\pi\csc^2(\pi z)(z+1/2)^2 - 2(z+1/2)\cot(\pi z)}{16(z+1/2)^4} \\
&= \lim_{z\to 1/2} \frac{-\pi\csc^2(\pi z)(z+1/2) - 2\cot(\pi z)}{16(z+1/2)^3} \\
&= \frac{-\pi\csc^2(\pi/2)(1) - 2\cot(\pi/2)}{16(1)^4} \\
&= -\frac{\pi}{16}.
\end{aligned}
$$

After similar calculations, we get

$$\operatorname{Res}(f(z)\cot(\pi z), -1/2) = -\frac{\pi}{16}.$$

Hence, we get

$$\sum_{k=-\infty}^{\infty} \frac{1}{(4k^2-1)^2} = -\pi\Big(-\frac{\pi}{16} - \frac{\pi}{16}\Big) = \frac{\pi^2}{8}.$$