COUNTING INVOLUTIONS ON MULTICOMPLEX SPACES

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Abstract. Motivated by work on generalized Cauchy–Riemann equations, we describe the set of involutions of the multicomplex numbers that preserve imaginary units. We count these involutions by translating the problem into the language of combinatorial matrix theory. In passing, we obtain new results on multicomplex numbers that are surprisingly unknown in the literature.

Removing the condition of preserving imaginary units, we find a formula for the number of r-involutions on multicomplex spaces. Our proof is based on the construction of a bijection between real-linear automorphisms of the multicomplex numbers of order n and the set of signed permutations of length 2^{n-1}.

1. Introduction

Let f be a function on an associative real algebra A with multiplicative identity. We say that f is an involution of A if f is a real-linear automorphism satisfying $f(f(a)) = a$ for any $a \in A$. The usual definition of an involution involves only the condition $f(f(a)) = a$. However, for the quaternions, bicomplex numbers, and general algebras over commutative fields (see [8], [10], [14]), the above definition was adopted. Therefore, to be consistent with these references, we will adopt this definition.

When A is the field of quaternions, we know from [8, 10] that there are infinitely many involutions. If $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ is a quaternion with the usual rules

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}$$

and $\mathbf{i^2} = \mathbf{j^2} = \mathbf{k^2} = -1$, then any involution is given by $f_\mu(q) = \mu q \mu$, where $\mu = a_0\mathbf{i} + b_0\mathbf{j} + c_0\mathbf{k}$ with $a_0^2 + b_0^2 + c_0^2 = 1$. For other real algebras, however, the situation might change drastically.

In a recent note [14], the author replaced quaternions by the commutative ring of bicomplex numbers. A bicomplex number s is defined as

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$s = a + b\mathbf{i_1} + c\mathbf{i_2} + d\mathbf{i_1}\mathbf{i_2}$ with the rules

$$\mathbf{i_1}\mathbf{i_2} = \mathbf{i_2}\mathbf{i_1}, \quad \mathbf{i_1^2} = \mathbf{i_2^2} = -1.$$

The set of bicomplex numbers is usually denoted by $\mathbb{MC}(2)$ or $\mathbb{BC}$. Main references for these are [12, 15]. From [14, Theorem 1], we know there are six involutions of $\mathbb{MC}(2)$. This result contrasts with the similar one obtained for quaternions and therefore makes the set of bicomplex numbers akin to the complex numbers, where the only involutions are $z \mapsto z$ and $z \mapsto \bar{z}$.

The goal of this paper is to extend the result from [14] to the multicomplex numbers of order $n \geq 1$, denoted by $\mathbb{MC}(n)$. The multicomplex numbers are a generalization of the complex numbers and the bicomplex numbers to higher dimensions. Section 2 gives some preliminaries on the multicomplex numbers.

Our original motivation for this problem comes from [20], where the authors used a specific class of involutions to obtain Cauchy–Riemann equations characterizing multicomplex holomorphic functions. Therefore, the present work has potential applications to hypercomplex analysis and could be used to describe classes of functions other than the family of multicomplex holomorphic functions through new types of Cauchy–Riemann equations.

To state our first main result, we need to introduce some notations. We let $\mathbf{i_1}, \ldots, \mathbf{i_n}$ be the elementary commuting imaginary units of $\mathbb{MC}(n)$ and define the set $\mathbb{I}(n)$ as the set of numbers that can be written as $\mathbf{i_1}^{a_1} \cdots \mathbf{i_n}^{a_n}$ with $a_k \in \{0, 1\}$. Observe that since $\mathbf{i_k^2} = -1$, $1 \leq k \leq n$, and the elementary units commute, the elements of $\mathbb{I}(n)$ square to ± 1. Our present investigation highlighted a surprising phenomenon in the multicomplex numbers: there are numbers squaring to ± 1 that are not in the set of units $\mathbb{I}(n)$ (see Section 2 for more details on this). In other words, we discovered new square roots of 1 and -1 in the space of multicomplex numbers.

This last phenomenon is not present in the set of bicomplex numbers, and therefore every involution obtained in [14] maps every element of $\mathbb{I}(2)$ to an element of $\mathbb{I}(2)$. It is therefore natural to ask the following question: how many involutions of $\mathbb{MC}(n)$ send the units of $\mathbb{I}(n)$ to the units of $\mathbb{I}(n)$? We call such involutions $\mathbb{I}(n)$-preserving involutions. The following theorem answers this question.

Theorem 1.1. *The number of $\mathbb{I}(n)$-preserving involutions of $\mathbb{MC}(n)$, $n \geq 1$, is*

$$\sum_{k=\lceil n/2 \rceil}^{n} \Big(\prod_{j=1}^{k-1} \frac{2^n - 2^j}{2^k - 2^j} \Big) \Big(\prod_{j=0}^{n-k-1} (2^k - 2^j) \Big) 2^k,$$

where an empty product is understood to be equal to 1.

We could not find a reference for the sequence of values given by the formula in Theorem 1.1 on the On-Line Encyclopedia of Integer Sequences. The idea of the proof of Theorem 1.1 is to translate the original problem into a counting problem in combinatorial matrix theory. We were therefore led to count certain matrices with entries in $\{0, 1\}$ subject to precise constraints using linear algebra tools. The proof of the last theorem is presented in Section 3.

On the other hand, we note that counting involutions is very natural in many settings other than real algebras, where involutions play a fundamental role. Perhaps the most prominent example is counting involutions on the symmetric group S_n (see, for instance, [6, 13]).

In Section 4, we investigate a connection between signed permutations of length n and real-linear automorphisms of $\mathbb{MC}(n)$. A signed permutation of length n is a permutation of $\{1, 2, \ldots, n\}$ written in one-line notation where each entry may have a bar over it. For instance, $\pi = 3\bar{1}2$ is a signed permutation. We write B_n for the set of signed permutations of length n, which also corresponds to the group of symmetries of a hypercube, the hyperoctahedral group, which is a Coxeter group of type B and of rank n [5, 7, 16]. Precisely, we show the following result.

Theorem 1.2. *For each integer $n \geq 1$, there is a bijection between the set of real-linear automorphisms of $\mathbb{MC}(n)$ and $B_{2^{n-1}}$. Furthermore, this bijection sends the identity function to the identity signed permutation and is compatible with composition.*

The construction of the bijection in the last theorem is based on new characterizations of multicomplex numbers squaring to 1 and -1 that were surprisingly unknown in the literature.

As a consequence of Theorem 1.2, we obtain an exact formula for the number of involutions that are not necessarily $\mathbb{I}(n)$-preserving. The statement and the proof of this result are presented in Section 5. Lastly, we also work out a formula giving the number of r-involutions. Here, an r-involution is a real-linear automorphisms $f : \mathbb{MC}(n) \to \mathbb{MC}(n)$ such that $f^{(r)} = \mathrm{Id}$, where $f^{(r)}$ is the r-fold composition of f with r a positive integer and Id is the identity map. The proof of this result is presented in Section 6.

2. Background on multicomplex numbers

In 1892, Segre [19] introduced an algebraic structure that he called n-complex numbers with the goal of defining a multiplication operation between vectors of $\mathbb{C}^n$, for $n \geq 2$.

Interest in the theory of n-complex numbers (nowadays referred to as multicomplex numbers) and its applications have grown over the past decades. For example, they are used to introduce generalizations of concepts from real and complex analysis, e.g., multicomplex fractional operators [4], multicomplex hyperanalytic functions [22], Laurent series [11], Riemannian and semi-Riemannian geometry [23], and multicomplex holomorphic functions [20]. We also mention the use of multicomplex numbers to generalize the Mandelbrot set to higher dimensions [2, 3, 9, 17], in theoretical physics to generalize the linear and non-linear Schrödinger equation [18, 21], and in machine learning to generalize complex-valued neural networks [1].

A modern treatment of these numbers is presented in [15] with a preface describing the history of the development of associative algebras. We will mainly follow the presentation given in [2], with some slight changes in the notations.

2.1. **Multicomplex numbers.** The definition of the multicomplex numbers is given recursively. Let $\mathbb{MC}(0)$ be the set of real numbers and let $\mathbb{MC}(n)$, $n \geq 1$, be the set

$$\mathbb{MC}(n) := \{\eta = \eta_1 + \eta_2 \mathbf{i_n} \,:\, \eta_1, \eta_2 \in \mathbb{MC}(n-1),\, \mathbf{i_n^2} = -1\}. \tag{2.1}$$

For example, when $n = 1$, we obtain the set $\mathbb{MC}(1)$ of complex numbers $\eta_1 + \eta_2 \mathbf{i_1}$, where $\mathbf{i_1^2} = -1$. When $n = 2$, we obtain the set $\mathbb{MC}(2)$ of bicomplex numbers $\eta_1 + \eta_2 \mathbf{i_2}$, where η_1, η_2 are complex numbers, $\mathbf{i_2^2} = -1$, and $\mathbf{i_1} \neq \mathbf{i_2}$. We say that two multicomplex numbers η and ζ are equal if and only if $\eta_1 = \zeta_1$ and $\eta_2 = \zeta_2$. If we let $\eta_2 = 0$ in the expression of a multicomplex number $\eta = \eta_1 + \eta_2 \mathbf{i_n}$, we see that $\mathbb{MC}(n-1) \subset \mathbb{MC}(n)$.

The set of multicomplex numbers becomes a commutative ring if we endow it with the following algebraic operations:

1) $\eta + \zeta := (\eta_1 + \zeta_1) + (\eta_2 + \zeta_2)\mathbf{i_n}$;
2) $\eta\zeta := (\eta_1\zeta_1 - \eta_2\zeta_2) + (\eta_1\zeta_2 + \eta_2\zeta_1)\mathbf{i_n}$.

These last operations must be understood recursively.

Let $\eta = \eta_1 + \eta_2 \mathbf{i_n}$ be a multicomplex number. Then $\eta_1, \eta_2 \in \mathbb{MC}(n-1)$, so there are multicomplex numbers $\eta_{11}, \eta_{12}, \eta_{21}, \eta_{22} \in \mathbb{MC}(n-2)$ such that $\eta_1 = \eta_{11} + \eta_{12}\mathbf{i_{n-1}}$ and $\eta_2 = \eta_{21} + \eta_{22}\mathbf{i_{n-1}}$. Replacing the η_1 and η_2 in the expression for η, we obtain the representation of a multicomplex number $\zeta \in \mathbb{MC}(n)$ in terms of four components in $\mathbb{MC}(n-2)$,

$$\zeta = (\eta_{11} + \eta_{12}\mathbf{i_{n-1}}) + (\eta_{21} + \eta_{22}\mathbf{i_{n-1}})\mathbf{i_n}.$$

From the definition of the multiplication, we can distribute $\mathbf{i_n}$ to obtain

$$\zeta = \eta_{11} + \eta_{12}\mathbf{i_{n-1}} + \eta_{21}\mathbf{i_n} + \eta_{22}\mathbf{i_{n-1}}\mathbf{i_n}.$$

For example, a bicomplex number $\eta = \eta_1 + \eta_2 \mathbf{i_2}$ can be expressed as a linear combination involving four real coefficients,

$$\eta = \eta_{11} + \eta_{12}\mathbf{i_1} + \eta_{21}\mathbf{i_2} + \eta_{22}\mathbf{i_1}\mathbf{i_2}.$$

We can continue this process recursively until we reach the set $\mathbb{MC}(0)$. At each stage k $(1 \leq k \leq n)$ of the process, we obtain a representation of a multicomplex number in terms of 2^k multicomplex numbers in $\mathbb{MC}(n-k)$. All of these representations are called the canonical representation (or the Cartesian representation) of a multicomplex number. The canonical representation we are interested in is the one in terms of the 2^n components in $\mathbb{MC}(0)$. To be more explicit, recall that $\mathbb{I}(n)$ is the set of all different possible products of the elements in the set $\{1, \mathbf{i_1}, \mathbf{i_2}, \dots, \mathbf{i_n}\}$. Since the multiplication is commutative, the cardinality of $\mathbb{I}(n)$ is 2^n. Therefore, we can write any multicomplex number as

$$\eta = \sum_{\mathbf{i} \in \mathbb{I}(n)} \eta_{\mathbf{i}} \mathbf{i}, \tag{2.2}$$

where $\eta_{\mathbf{i}} \in \mathbb{R}$. This tells us that the elements of $\mathbb{I}(n)$ form a (real-)basis of $\mathbb{MC}(n)$. For instance, when $n = 2$ or 3, the following holds:

1) For $\eta \in \mathbb{MC}(2)$, we have

$$\eta = \eta_1 + \eta_{\mathbf{i_1}}\mathbf{i_1} + \eta_{\mathbf{i_2}}\mathbf{i_2} + \eta_{\mathbf{i_1}\mathbf{i_2}}\mathbf{i_1}\mathbf{i_2}.$$

2) For $\eta \in \mathbb{MC}(3)$, we have

$$\begin{aligned}\eta = \eta_1 &+ \eta_{\mathbf{i_1}}\mathbf{i_1} + \eta_{\mathbf{i_2}}\mathbf{i_2} + \eta_{\mathbf{i_1}\mathbf{i_2}}\mathbf{i_1}\mathbf{i_2} + \eta_{\mathbf{i_3}}\mathbf{i_3} \\ &+ \eta_{\mathbf{i_1}\mathbf{i_3}}\mathbf{i_1}\mathbf{i_3} + \eta_{\mathbf{i_2}\mathbf{i_3}}\mathbf{i_2}\mathbf{i_3} + \eta_{\mathbf{i_1}\mathbf{i_2}\mathbf{i_3}}\mathbf{i_1}\mathbf{i_2}\mathbf{i_3}.\end{aligned}$$

Using this representation and the algebraic operations defined above, we can view the set $\mathbb{MC}(n)$ as a commutative and associative algebra on the set of real numbers.

2.2. **An idempotent representation for multicomplex numbers.** Of particular importance in the set of multicomplex numbers are the numbers η such that $\eta^2 = \eta$, which are called idempotent numbers. In particular, we consider

$$\mathbf{e_n} := \frac{1 + \mathbf{i_{n-1}}\mathbf{i_n}}{2} \quad \text{and} \quad \overline{\mathbf{e}}_\mathbf{n} := \frac{1 - \mathbf{i_{n-1}}\mathbf{i_n}}{2}.$$

An additional property that these numbers have is that $\mathbf{e_n}\overline{\mathbf{e}}_\mathbf{n} = 0$. If we multiply a multicomplex number $\eta = \eta_1 + \eta_2 \mathbf{i_n}$ by $\mathbf{e_n}$ and by $\overline{\mathbf{e}}_\mathbf{n}$ respectively, we obtain

$$\eta\mathbf{e_n} = (\eta_1 - \eta_2\mathbf{i_{n-1}})\mathbf{e_n} \quad \text{and} \quad \eta\overline{\mathbf{e}}_\mathbf{n} = (\eta_1 + \eta_2\mathbf{i_{n-1}})\overline{\mathbf{e}}_\mathbf{n}.$$

Since $\mathbf{e_n} + \overline{\mathbf{e}}_\mathbf{n} = 1$, summing $\eta\mathbf{e_n}$ and $\eta\overline{\mathbf{e}}_\mathbf{n}$ yields the idempotent representation of a multicomplex number, namely

$$\eta = (\eta_1 - \eta_2\mathbf{i_{n-1}})\mathbf{e_n} + (\eta_1 + \eta_2\mathbf{i_{n-1}})\overline{\mathbf{e}}_\mathbf{n}.$$

We see that the numbers multiplying $\mathbf{e_n}$ and $\overline{\mathbf{e}}_\mathbf{n}$ are elements of $\mathbb{MC}(n-1)$, which we call the idempotent components of η. We will denote them by $\eta_{\mathbf{e_n}}$ and $\eta_{\overline{\mathbf{e}}_\mathbf{n}}$, respectively. The idempotent representation can therefore be rewritten as

$$\eta = \eta_{\mathbf{e_n}}\mathbf{e_n} + \eta_{\overline{\mathbf{e}}_\mathbf{n}}\overline{\mathbf{e}}_\mathbf{n}. \tag{2.3}$$

Note that two multicomplex numbers are equal if and only if their idempotent components are equal.

The idempotent representation is important because it transforms the multiplication of multicomplex numbers into a component-wise multiplication. More precisely, if $\eta = \eta_{\mathbf{e_n}}\mathbf{e_n} + \eta_{\overline{\mathbf{e}}_\mathbf{n}}\overline{\mathbf{e}}_\mathbf{n}$ and $\zeta = \zeta_{\mathbf{e_n}}\mathbf{e_n} + \zeta_{\overline{\mathbf{e}}_\mathbf{n}}\overline{\mathbf{e}}_\mathbf{n}$, then we have

$$\eta\zeta = \eta_{\mathbf{e_n}}\zeta_{\mathbf{e_n}}\mathbf{e_n} + \eta_{\overline{\mathbf{e}}_\mathbf{n}}\zeta_{\overline{\mathbf{e}}_\mathbf{n}}\overline{\mathbf{e}}_\mathbf{n}. \tag{2.4}$$

We now apply this result to the idempotent components of a multicomplex number η. Define

$$\mathbf{e_{n-1}} := \frac{1 + \mathbf{i_{n-2}}\mathbf{i_{n-1}}}{2} \quad \text{and} \quad \overline{\mathbf{e}}_\mathbf{n-1} := \frac{1 - \mathbf{i_{n-2}}\mathbf{i_{n-1}}}{2}.$$

Then, the idempotent components $\eta_{\mathbf{e_n}}$ and $\eta_{\overline{\mathbf{e}}_\mathbf{n}}$ of $\eta \in \mathbb{MC}(n)$ can be written as

$$\eta_{\mathbf{e_n}} = \eta_{\mathbf{e_{n-1}}\mathbf{e_n}}\mathbf{e_{n-1}} + \eta_{\overline{\mathbf{e}}_\mathbf{n-1}\mathbf{e_n}}\overline{\mathbf{e}}_\mathbf{n-1}$$

and

$$\eta_{\overline{\mathbf{e}}_\mathbf{n}} = \eta_{\mathbf{e_{n-1}}\overline{\mathbf{e}}_\mathbf{n}}\mathbf{e_{n-1}} + \eta_{\overline{\mathbf{e}}_\mathbf{n-1}\overline{\mathbf{e}}_\mathbf{n}}\overline{\mathbf{e}}_\mathbf{n-1},$$

where $\eta_{\mathbf{e_{n-1}}\mathbf{e_n}}, \eta_{\overline{\mathbf{e}}_\mathbf{n-1}\mathbf{e_n}}, \eta_{\mathbf{e_{n-1}}\overline{\mathbf{e}}_\mathbf{n}}, \eta_{\overline{\mathbf{e}}_\mathbf{n-1}\overline{\mathbf{e}}_\mathbf{n}} \in \mathbb{MC}(n-2)$. Replacing these in the idempotent representation of $\eta \in \mathbb{MC}(n)$, we obtain a second idempotent representation in terms of components in $\mathbb{MC}(n-2)$,

$$\eta = \eta_{\mathbf{e_{n-1}}\mathbf{e_n}}\mathbf{e_{n-1}}\mathbf{e_n} + \eta_{\overline{\mathbf{e}}_\mathbf{n-1}\mathbf{e_n}}\overline{\mathbf{e}}_\mathbf{n-1}\mathbf{e_n} + \eta_{\mathbf{e_{n-1}}\overline{\mathbf{e}}_\mathbf{n}}\mathbf{e_{n-1}}\overline{\mathbf{e}}_\mathbf{n} + \eta_{\overline{\mathbf{e}}_\mathbf{n-1}\overline{\mathbf{e}}_\mathbf{n}}\overline{\mathbf{e}}_\mathbf{n-1}\overline{\mathbf{e}}_\mathbf{n}.$$

More generally, define the following elements for each integer $k \geq 2$:

$$\mathbf{e_k} := \frac{1 + \mathbf{i_{k-1}}\mathbf{i_k}}{2} \quad \text{and} \quad \overline{\mathbf{e}}_\mathbf{k} := \frac{1 - \mathbf{i_{k-1}}\mathbf{i_k}}{2}.$$

We then define a family of sets $\mathcal{E}(k, n)$ inductively for $n \geq 2$ and $2 \leq k \leq n$:

1) $\mathcal{E}(n, n) := \{\mathbf{e_n}, \overline{\mathbf{e}}_\mathbf{n}\}$ for $k = n$;
2) $\mathcal{E}(k, n) := \mathcal{E}(k+1, n)\mathbf{e_k} \cup \mathcal{E}(k+1, n)\overline{\mathbf{e}}_\mathbf{k}$ for $2 \leq k < n$.

Now, for any $2 \leq k \leq n$, an induction argument shows that the cardinality of $\mathcal{E}(k,n)$ is 2^{n-k+1}. Also, by induction, we have that if $\boldsymbol{\varepsilon} \in \mathcal{E}(k,n)$, then $\boldsymbol{\varepsilon}^2 = \boldsymbol{\varepsilon}$, and if $\boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathcal{E}(k,n)$ with $\boldsymbol{\varepsilon}_1 \neq \boldsymbol{\varepsilon}_2$, then $\boldsymbol{\varepsilon}_1\boldsymbol{\varepsilon}_2 = 0$.

Finally, any multicomplex number $\eta \in \mathbb{MC}(n)$ can be rewritten as

$$\eta = \sum_{\boldsymbol{\varepsilon} \in \mathcal{E}(k,n)} \eta_{\boldsymbol{\varepsilon}} \boldsymbol{\varepsilon},$$

where $\eta_{\boldsymbol{\varepsilon}} \in \mathbb{MC}(k-1)$ for all $\boldsymbol{\varepsilon} \in \mathcal{E}(k,n)$. The special case when $k = 2$ will be of particular importance to us. For this reason, we let $\mathcal{E}_n := \mathcal{E}(2,n)$ and therefore any $\eta \in \mathbb{MC}(n)$ can be written as

$$\eta = \sum_{\boldsymbol{\varepsilon} \in \mathcal{E}_n} \eta_{\boldsymbol{\varepsilon}} \boldsymbol{\varepsilon}, \tag{2.5}$$

where $\eta_{\boldsymbol{\varepsilon}} \in \mathbb{MC}(1)$ for all $\boldsymbol{\varepsilon} \in \mathcal{E}_n$.

These new idempotent representations still have the advantage of simplifying the operation of multiplication. If

$$\eta = \sum_{\boldsymbol{\varepsilon} \in \mathcal{E}(k,n)} \eta_{\boldsymbol{\varepsilon}} \boldsymbol{\varepsilon} \quad \text{and} \quad \zeta = \sum_{\boldsymbol{\varepsilon} \in \mathcal{E}(k,n)} \zeta_{\boldsymbol{\varepsilon}} \boldsymbol{\varepsilon},$$

then the following holds:

1) $\eta = \zeta$ if and only if $\eta_{\boldsymbol{\varepsilon}} = \zeta_{\boldsymbol{\varepsilon}}$ for all $\boldsymbol{\varepsilon} \in \mathcal{E}(k,n)$;
2) $\eta + \zeta = \sum_{\boldsymbol{\varepsilon} \in \mathcal{E}(k,n)} (\eta_{\boldsymbol{\varepsilon}} + \zeta_{\boldsymbol{\varepsilon}})\boldsymbol{\varepsilon}$;
3) $\eta\zeta = \sum_{\boldsymbol{\varepsilon} \in \mathcal{E}(k,n)} (\eta_{\boldsymbol{\varepsilon}}\zeta_{\boldsymbol{\varepsilon}})\boldsymbol{\varepsilon}$.

2.3. **Representation theorems and bijections.** We use the notation U_n to denote the set of multicomplex numbers squaring to -1 and H_n for numbers squaring to 1, namely

$$U_n := \{\eta \in \mathbb{MC}(n) \,:\, \eta^2 = -1\} \quad \text{and} \quad H_n := \{\eta \in \mathbb{MC}(n) \,:\, \eta^2 = 1\}.$$

We also write E_n for the set of idempotent elements of $\mathbb{MC}(n)$, that is,

$$E_n := \{\eta \in \mathbb{MC}(n) \,:\, \eta^2 = \eta\}.$$

For example, U_3 contains the numbers

1) $\mathbf{i_1}$, $-\mathbf{i_1}$;
2) $\mathbf{i_2}$, $-\mathbf{i_2}$;
3) $\mathbf{i_3}$, $-\mathbf{i_3}$;
4) $\mathbf{i_1 i_2 i_3}$, $-\mathbf{i_1 i_2 i_3}$;
5) $\frac{\mathbf{i_1+i_2+i_3+i_1i_2i_3}}{2}$, $-\frac{\mathbf{i_1+i_2+i_3+i_1i_2i_3}}{2}$;
6) $\frac{\mathbf{i_1-i_2-i_3+i_1i_2i_3}}{2}$, $-\frac{\mathbf{i_1-i_2-i_3+i_1i_2i_3}}{2}$;
7) $\frac{\mathbf{i_1+i_2-i_3-i_1i_2i_3}}{2}$, $-\frac{\mathbf{i_1+i_2-i_3-i_1i_2i_3}}{2}$;
8) $\frac{\mathbf{i_1-i_2+i_3-i_1i_2i_3}}{2}$, $-\frac{\mathbf{i_1-i_2+i_3-i_1i_2i_3}}{2}$,

while H_3 contains

1) $1,\ -1$;
2) $\mathbf{i_1 i_2},\ -\mathbf{i_1 i_2}$;
3) $\mathbf{i_1 i_3},\ -\mathbf{i_1 i_3}$;
4) $\mathbf{i_2 i_3},\ -\mathbf{i_2 i_3}$;
5) $\frac{1+\mathbf{i_1i_2}+\mathbf{i_1i_3}+\mathbf{i_2i_3}}{2},\ -\frac{1+\mathbf{i_1i_2}+\mathbf{i_1i_3}+\mathbf{i_2i_3}}{2}$;
6) $\frac{1-\mathbf{i_1i_2}-\mathbf{i_1i_3}+\mathbf{i_2i_3}}{2},\ -\frac{1-\mathbf{i_1i_2}-\mathbf{i_1i_3}+\mathbf{i_2i_3}}{2}$;
7) $\frac{1+\mathbf{i_1i_2}-\mathbf{i_1i_3}-\mathbf{i_2i_3}}{2},\ -\frac{1+\mathbf{i_1i_2}-\mathbf{i_1i_3}-\mathbf{i_2i_3}}{2}$;
8) $\frac{1-\mathbf{i_1i_2}+\mathbf{i_1i_3}-\mathbf{i_2i_3}}{2},\ -\frac{1-\mathbf{i_1i_2}+\mathbf{i_1i_3}-\mathbf{i_2i_3}}{2}$,

and E_3 contains

1) $1,\ 0$;
2) $\frac{1+\mathbf{i_1i_2}}{2},\ \frac{1-\mathbf{i_1i_2}}{2}$;
3) $\frac{1+\mathbf{i_1i_3}}{2},\ \frac{1-\mathbf{i_1i_3}}{2}$;
4) $\frac{1+\mathbf{i_2i_3}}{2},\ \frac{1-\mathbf{i_2i_3}}{2}$;
5) $\frac{3+\mathbf{i_1i_2}+\mathbf{i_1i_3}+\mathbf{i_2i_3}}{4},\ \frac{1-\mathbf{i_1i_2}-\mathbf{i_1i_3}-\mathbf{i_2i_3}}{4}$;
6) $\frac{3-\mathbf{i_1i_2}-\mathbf{i_1i_3}+\mathbf{i_2i_3}}{4},\ \frac{1+\mathbf{i_1i_2}+\mathbf{i_1i_3}-\mathbf{i_2i_3}}{4}$;
7) $\frac{3+\mathbf{i_1i_2}-\mathbf{i_1i_3}-\mathbf{i_2i_3}}{4},\ \frac{1-\mathbf{i_1i_2}+\mathbf{i_1i_3}+\mathbf{i_2i_3}}{4}$;
8) $\frac{3-\mathbf{i_1i_2}+\mathbf{i_1i_3}-\mathbf{i_2i_3}}{4},\ \frac{1+\mathbf{i_1i_2}-\mathbf{i_1i_3}+\mathbf{i_2i_3}}{4}$.

Also note that the set E_n contains many more elements than the set $\mathcal{E}_n$. For example, the elements of $\mathcal{E}_3$ correspond to the second entry in items 5)–8) in the list of elements of E_3.

The above examples suggest that there is a bijection between U_n and H_n. It can be given explicitly as follows.

Proposition 2.1. *Let u be any element of U_n. We have*

$$U_n = uH_n \quad \text{and} \quad H_n = uU_n.$$

Proof. Fix $u \in U_n$. Define the map $f : H_n \to U_n$ by $f(h) := uh$. Then f is injective. If $v \in U_n$, then $-uv \in H_n$ because $(-uv)^2 = 1$. Therefore, there is an $h \in H_n$ such that $h = -uv$ and so $uh = v$. This means that f is surjective and therefore f is a bijection. Since $f(H_n) = U_n$, we obtain the first equality. The second equality is obtained similarly. □

We are now ready to prove the following characterization of multicomplex numbers squaring to ± 1.

Proposition 2.2. *For each integer $n \geq 1$, there are $2^{2^{n-1}}$ multicomplex numbers squaring to 1 and $2^{2^{n-1}}$ squaring to -1.*

Proof. By Proposition 2.1, it suffices to prove it for U_n. We proceed by induction on n. For $n = 1$, the set $\mathbb{MC}(1)$ is the set of complex numbers and there are only two solutions to $\eta^2 = -1$, namely $\mathbf{i_1}$ and $-\mathbf{i_1}$. This is exactly $2^{2^{n-1}}$ for $n = 1$. Suppose that there are $2^{2^{n-1}}$ solutions for $\eta^2 = -1$ in $\mathbb{MC}(n)$. We will show that there are 2^{2^n} solutions for $\eta^2 = -1$ in $\mathbb{MC}(n+1)$. Let $\eta \in \mathbb{MC}(n+1)$ be written in its idempotent representation, that is, $\eta = \eta_{\mathbf{e_{n+1}}} \mathbf{e_{n+1}} + \eta_{\overline{\mathbf{e}}_{\mathbf{n+1}}} \overline{\mathbf{e}}_{\mathbf{n+1}}$, where $\eta_{\mathbf{e_{n+1}}}, \eta_{\overline{\mathbf{e}}_{\mathbf{n+1}}} \in \mathbb{MC}(n)$. Then, according to (2.4), η is a solution to the equation $\eta^2 = -1$ if and only if $(\eta_{\mathbf{e_{n+1}}}, \eta_{\overline{\mathbf{e}}_{\mathbf{n+1}}})$ is a solution to the system of equations

$$\begin{cases}\eta_{\mathbf{e_{n+1}}}^2 = -1,\\ \eta_{\mathbf{\overline{e}_{n+1}}}^2 = -1.\end{cases}$$

Since $\eta_{\mathbf{e_{n+1}}}, \eta_{\mathbf{\overline{e}_{n+1}}} \in \mathbb{MC}(n)$, by the induction hypothesis there are $2^{2^{n-1}}$ solutions to each equation in the system. Therefore, there are $2^{2^{n-1}} \cdot 2^{2^{n-1}} = 2^{2^n}$ solutions to $\eta^2 = -1$ in $\mathbb{MC}(n+1)$. This ends the induction and the claim is proved. □

It is possible to prove a similar statement for the set of multicomplex number η satisfying the equation $\eta^2 = \eta$. Instead, we give an explicit bijection between H_n and E_n to prove the next proposition.

Proposition 2.3. *For each integer $n \geq 1$, there are $2^{2^{n-1}}$ idempotent elements in $\mathbb{MC}(n)$.*

Proof. If $h \in H_n$, then

$$\left(\frac{1+h}{2}\right)^2 = \frac{1+h}{2}.$$

It follows that the map $h \mapsto (1+h)/2$ is well defined. It is also clearly a bijection. □

An interesting corollary to the proofs of Proposition 2.2 and Proposition 2.3 is the following representation theorem for certain multicomplex numbers.

Corollary 2.4. *Let $\eta \in \mathbb{MC}(n)$.*

(i) If $\eta^2 = -1$, then η can be written as

$$\eta = \sum_{\boldsymbol{\varepsilon} \in \mathcal{E}_n} \eta_{\boldsymbol{\varepsilon}} \boldsymbol{\varepsilon}$$

where the components $\eta_{\boldsymbol{\varepsilon}} \in \{\mathbf{i_1}, -\mathbf{i_1}\}$.

(ii) If $\eta^2 = 1$, then η can be written as

$$\eta = \sum_{\boldsymbol{\varepsilon} \in \mathcal{E}_n} \eta_{\boldsymbol{\varepsilon}} \boldsymbol{\varepsilon}$$

where the components $\eta_{\boldsymbol{\varepsilon}} \in \{1, -1\}$.

(iii) If $\eta^2 = \eta$, then η can be written as

$$\eta = \sum_{\boldsymbol{\varepsilon} \in \mathcal{E}_n} \eta_{\boldsymbol{\varepsilon}} \boldsymbol{\varepsilon}$$

where the components $\eta_{\boldsymbol{\varepsilon}} \in \{0, 1\}$.

Proof. The result for numbers squaring to -1 is immediate. Then Part (ii) follows from Proposition 2.1. Part (iii) follows from Proposition 2.3 and the fact that $1 = \sum_{\boldsymbol{\varepsilon} \in \mathcal{E}_n} \boldsymbol{\varepsilon}$. □

3. Involutions preserving elementary units

We start by giving a precise definition of what we mean by an involution on the set $\mathbb{MC}(n)$.

Definition 3.1. A function $f : \mathbb{MC}(n) \to \mathbb{MC}(n)$ is said to be an *involution* if the following conditions are satisfied:

(a) $f(f(\eta)) = \eta$ for any $\eta \in \mathbb{MC}(n)$;
(b) $f(\eta+\zeta) = f(\eta)+f(\zeta)$ and $f(\lambda\eta) = \lambda f(\eta)$ for any $\eta, \zeta \in \mathbb{MC}(n)$ and $\lambda \in \mathbb{R}$;
(c) $f(\eta\zeta) = f(\eta)f(\zeta)$ for any $\eta, \zeta \in \mathbb{MC}(n)$.

If we take a closer look at our definition of the term "involution", we require that the function is a real-linear homomorphism which is its own inverse. When we only require that f is invertible but not necessarily its own inverse, we shall only say that f is a real-linear automorphism of $\mathbb{MC}(n)$.

Our focus is now on proving Theorem 1.1. The method of proof will be quite different from the next sections. This comes from the fact that it is hard to devise a workable condition to detect if an $h \in H_n$ is an element of $\mathbb{I}(n)$ based on the idempotent components of h. For this section, it will be more useful to use the canonical representation (2.2) of a multicomplex number.

If f is an involution, then for any multicomplex number η, we have

$$f(\eta) = \sum_{\mathbf{i}\in\mathbb{I}(n)} \eta_{\mathbf{i}} f(\mathbf{i}).$$

Since each $\mathbf{i} \in \mathbb{I}(n)\backslash\{1\}$ is a product of the units $\mathbf{i_1}, \mathbf{i_2}, \ldots, \mathbf{i_n}$ and f is a real-linear homomorphism, we obtain the following proposition.

Proposition 3.2. *If $f : \mathbb{MC}(n) \to \mathbb{MC}(n)$ is an involution, then its values are completely determined by its action on $\mathbf{i_1}, \mathbf{i_2}, \ldots, \mathbf{i_n}$.*

Now, what are the possible values of each $f(\mathbf{i_k})$ for $1 \le k \le n$? Since $f(\mathbf{i_k}) \in U_n$ and here we restrict our attention to involutions preserving $\mathbb{I}(n)$, $f(\mathbf{i_k})$ should be a product of an odd number of imaginary units $\mathbf{i_1}$, $\mathbf{i_2}$, …, $\mathbf{i_n}$. Therefore, an $\mathbb{I}(n)$-preserving involution f of $\mathbb{MC}(n)$ can be characterized by

$$f(\mathbf{i_j}) = \mathbf{i_1}^{a_{1,j}}\mathbf{i_2}^{a_{2,j}}\cdots\mathbf{i_n}^{a_{n,j}}(-1)^{a_{n+1,j}}, \qquad 1 \le j \le n.$$

Furthermore, for $1 \le j \le n$ we have

$$-1 = f(-1) = f(\mathbf{i_j^2}) = f(\mathbf{i_j})^2 = (-1)^{a_{1,j}}(-1)^{a_{2,j}}\cdots(-1)^{a_{n,j}},$$

which implies

$$\sum_{k=1}^{n} a_{k,j} \equiv 1 \pmod 2, \qquad 1 \le j \le n.$$

The first equality in the above chain of equalities comes from the fact that f is, in particular, a ring homomorphism. In summary, we are trying to find the number of functions $f : \mathbb{MC}(n) \to \mathbb{MC}(n)$ satisfying the following conditions:

1) $f(-1) = -1$;
2) $\mathbf{i_j^2} = -1$ for $1 \le j \le n$;
3) $f(\mathbf{i_j}\mathbf{i_k}) = f(\mathbf{i_j})f(\mathbf{i_k})$ for $1 \le j, k \le n$;
4) $f(f(\mathbf{i_j})) = \mathbf{i_j}$ for $1 \le j \le n$;
5) $f(\mathbf{i_j}) = \mathbf{i_1}^{a_{1,j}}\mathbf{i_2}^{a_{2,j}} \cdots \mathbf{i_n}^{a_{n,j}}(-1)^{a_{n+1,j}}$ for $1 \le j \le n$;
6) $a_{1,j}, \dots, a_{n+1,j} \in \{0, 1\}$ and $\sum_{k=1}^{n} a_{k,j} \equiv 1 \pmod 2$ for $1 \le j \le n$.

With this setup, we are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. We will use a matrix representation to obtain the number of involutions satisfying the previous description. A function f as described by 1) to 6) above will be represented by the matrix

$$A_f := \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} & 0 \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} & 0 \\ a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,n} & 1 \end{bmatrix}.$$

The condition $\sum_{k=1}^{n} a_{k,j} \equiv 1 \pmod 2$ for $1 \le j \le n$ then translates to a condition on the matrix A_f as

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} & 0 \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n} & 0 \\ a_{n+1,1} & a_{n+1,2} & \dots & a_{n+1,n} & 1 \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \pmod 2. \tag{3.1}$$

To simplify the notation, we will simply write A for A_f. On the other hand, the condition $f(f(\mathbf{i_j})) = \mathbf{i_j}$, $1 \le j \le n$, translates into

$$A^2 \equiv I \pmod 2. \tag{3.2}$$

Our problem thus becomes the problem of enumerating matrices A with $\{0, 1\}$ entries and satisfying conditions (3.1) and (3.2).

We set $X = A - I$. Working with modulo 2, the equation $A^2 \equiv I \pmod 2$ is equivalent to $X^2 \equiv 0 \pmod 2$. The problem now becomes enumerating $(n+1) \times (n+1)$ matrices X such that

1) The entries of X are equal to 0 or 1;
2) $X^2 \equiv 0 \pmod 2$;

3) The sum of each column of X is $\equiv 0 \pmod 2$;
4) The right column of X has only zeros.

We denote by Y the submatrix of X obtained by omitting the right column of X and its bottom row, that is,

$$Y = \begin{bmatrix} a_{1,1}-1 & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2}-1 & \dots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \dots & a_{n,n}-1 \end{bmatrix}.$$

The conditions on X imply the following conditions on Y:

1) The entries of Y are equal to 0 or 1;
2) $Y^2 \equiv 0 \pmod 2$;
3) The sum of each column of Y is $\equiv 0 \pmod 2$.

We denote by k the dimension of the kernel of Y, that is, $k := \dim(\ker(Y))$. Because $Y^2 = 0$, we have

$$k \geq n/2. \tag{3.3}$$

Observe that the dimension of the kernel of Y is equal to the dimension of the kernel of $Y^\top$. It will be easier to work with this transpose.

We use the notation $\vec{e} := (1, 1, \dots, 1)^\top$. Condition 3 on the matrix Y is equivalent to

$$\vec{e} \in \ker(Y^\top).$$

For a fixed value of k, the number of ways of choosing $\ker(Y^\top)$ with the restriction that $\vec{e} \in \ker(Y^\top)$ is given by

$$B(k,n) := \prod_{j=1}^{k-1} \frac{2^n - 2^j}{2^k - 2^j}, \tag{3.4}$$

with the convention that $B(k,n) = 1$ for $k = 0, 1$. To see this, first note that the number of ways of choosing an ordered sequence of k linearly independent vectors (with $\vec{e}$ as the first vector of the sequence) is given by

$$\prod_{j=1}^{k-1} (2^n - 2^j), \tag{3.5}$$

since when choosing a new vector, one cannot choose any linear combination of previously chosen vectors. Now, many choices of vector sequences (or basis choices) will describe the same subspace. Given a basis of linearly independent vectors, the number of ways of choosing a basis that will span the same subspace (under the condition that $\vec{e}$ is the first vector of the ordered basis) is given by

$$\prod_{j=1}^{k-1}(2^k - 2^j). \tag{3.6}$$

Equality (3.4) follows from (3.5) and (3.6).

Now, suppose that the kernel $\ker(Y^\top)$ has been chosen. Let k again be the dimension of the kernel of $Y^\top$. Let $\vec{u}_1, \vec{u}_2, \ldots \vec{u}_k$ be a vector basis of $\ker(Y^\top)$. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n-k}$ be vectors such that $\vec{u}_1, \vec{u}_2, \ldots \vec{u}_k, \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n-k}$ is a vector basis of $\mathbb{Z}_2^n$. We will now find the number of ways of choosing $\mathrm{Ran}\,(Y^\top)$, the range of $Y^\top$.

Since $Y^2 \equiv 0 \pmod 2$, we deduce that $(Y^\top)^2 \equiv 0 \pmod 2$. This last identity implies that

$$(Y^\top)^2\vec{v}_j = \vec{0}, \qquad 1 \le j \le n-k,$$

and thus

$$Y^\top\vec{v}_j \in \ker(Y^\top), \qquad 1 \le j \le n-k.$$

We therefore have

$$Y^\top\vec{v}_j = \sum_{s=1}^{k} r_{s,j}\vec{u}_s,$$

where $r_{s,j} \in \{0,1\}$ and $1 \le j \le n-k$. The number of ways to choose the value of $Y^\top\vec{v}_1$ is given by

$$2^k - 1.$$

This comes from the fact that $\vec{v}_1 \notin \ker(Y^\top)$ and therefore the $r_{s,j}$ cannot all be zero. One can now choose the values of $Y^\top\vec{v}_2, Y^\top\vec{v}_3, \ldots, Y^\top\vec{v}_{n-k}$ under the restriction that the vectors $Y^\top\vec{v}_j$ must be linearly independent. To see why these vectors must be linearly independent, suppose that $\vec{w}$ is a linear combination of the vectors $\vec{v}_j$ such that $Y^\top\vec{w} = \vec{0}$. This implies $\vec{w} \in \ker(Y^\top)$. We thus have exhibited a vector $\vec{w}$ that can be expressed both as a linear combination of the vectors $\vec{u}_j$ and as a linear combination of the vectors $\vec{v}_j$. This contradicts the fact that $\vec{u}_1, \vec{u}_2, \ldots \vec{u}_k, \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_{n-k}$ is a basis of $\mathbb{Z}_2^n$. This constraint of linear independence implies that the number of ways of choosing the vector $\vec{v}_j$ is equal to $2^k - 2^{j-1}$ for $1 \le j \le n-k$. We conclude that the number of ways of choosing the image of $Y^\top$ is given by

$$D(k,n) = \prod_{j=0}^{n-k-1}(2^k - 2^j). \tag{3.7}$$

Putting everything together, the number of ways of choosing the matrix $Y^\top$, or equivalently the number of ways of choosing the matrix Y, is given by

$$B(k,n)D(k,n) = \prod_{j=1}^{k-1} \frac{2^n - 2^j}{2^k - 2^j} \prod_{j=0}^{n-k-1} (2^k - 2^j).$$

Finally, to fully specify the matrix X, one has to specify its bottom row (describing the signs in the involution). Let x_1, x_2, $\ldots$, x_n, x_{n+1} be the rows of the matrix X. The condition $X^2 \equiv 0 \pmod 2$ can be written, after transposing, as $(X^\top)^2 \equiv 0 \pmod 2$. Therefore, we obtain the following conditions:

$$X^\top x_j^\top \equiv 0 \pmod 2$$

for every $j = 1, 2, \ldots, n+1$. In particular, with $j = n+1$, we get $X^\top x_{n+1}^\top \equiv 0 \pmod 2$, which can be rewritten as

$$\begin{bmatrix} a_{1,1} - 1 & a_{2,1} & \ldots & a_{n,1} \\ a_{1,2} & a_{2,2} - 1 & \ldots & a_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} & a_{2,n} & \ldots & a_{n,n} - 1 \end{bmatrix} \begin{bmatrix} a_{n+1,1} \\ a_{n+1,2} \\ \vdots \\ a_{n+1,n} \end{bmatrix} \equiv 0 \pmod 2.$$

Notice that the matrix in the previous equation is precisely $Y^\top$. Hence the vector

$$\vec{w} := \begin{bmatrix} a_{n+1,1} \\ a_{n+1,2} \\ \vdots \\ a_{n+1,n} \end{bmatrix}$$

must be in the kernel of the matrix $Y^\top$. If the dimension of the kernel of $Y^\top$ is equal to k, then the number of ways of choosing the components of $\vec{w}$ is equal to

$$2^k. \tag{3.8}$$

This last number comes from the fact that $\vec{w} = \sum_{s=1}^k r_s \vec{u}_k$ with $r_s \in \{0, 1\}$.

From equations (3.3), (3.4), (3.7), and (3.8), we conclude that the number of involutions satisfying (1) to (6) is equal to

$$\sum_{n/2 \le k \le n} D(k,n)B(k,n)2^k = \sum_{n/2 \le k \le n} \Big(\prod_{j=1}^{k-1} \frac{2^n - 2^j}{2^k - 2^j} \Big) \Big(\prod_{j=0}^{n-k-1} 2^k - 2^j \Big) 2^k.$$

This completes the proof. □

3.1. **An algorithm.** The proof of Theorem 1.1 suggests a way to generate a list of $\mathbb{I}(n)$-preserving involutions for a fixed value of n.

1) Fix a value of k and loop over the values of $k \in [n/2, n]$.
2) Generate a basis $\vec{u}_1, \dots \vec{u}_k$ (including $\vec{e}$) of all subspaces of dimension k of $\mathbb{Z}_2^n$.
3) For each basis in the list generated in step 2, find a set of vectors $\vec{v}_1, \dots, \vec{v}_{n-k}$ so that the vectors $\vec{u}_1, \dots, \vec{u}_k, \vec{v}_1, \dots, \vec{v}_{n-k}$ form a basis of $\mathbb{Z}_2^n$.
4) For each vector $\vec{v}_j$, choose the image of $\vec{v}_j$ as a linear combination of the vectors $\vec{u}_j$. We denote this image by $\vec{s}_j$.
5) Obtain the matrix $Y^\top$ as in the above proof by solving
$$Y^\top[\vec{u}_1, \dots, \vec{u}_k, \vec{v}_1, \dots, \vec{v}_{n-k}] = [\vec{0}, \dots, \vec{0}, \vec{s}_1, \dots, \vec{s}_{n-k}].$$
6) Generate a list of involutions by looping over all possibilities for the choice of the sign vector associated to Y, fully specifying in the matrix X in the above proof.

Here is a sample of $\mathbb{I}(3)$-preserving involutions. To the best of the authors' knowledge, these involutions do not appear in any of the references using such involutions in their work.

1) $f(\eta) = \eta_1 - \eta_{\mathbf{i_1}}\mathbf{i_1} - \eta_{\mathbf{i_2}}\mathbf{i_2} + \eta_{\mathbf{i_1i_2}}\mathbf{i_1i_2} + \eta_{\mathbf{i_1i_2i_3}}\mathbf{i_3} + \eta_{\mathbf{i_2i_3}}\mathbf{i_1i_3} + \eta_{\mathbf{i_1i_3}}\mathbf{i_2i_3} + \eta_{\mathbf{i_3}}\mathbf{i_1i_2i_3}$;
2) $f(\eta) = \eta_1 - \eta_{\mathbf{i_1}}\mathbf{i_1} - \eta_{\mathbf{i_2}}\mathbf{i_2} + \eta_{\mathbf{i_1i_2}}\mathbf{i_1i_2} - \eta_{\mathbf{i_1i_2i_3}}\mathbf{i_3} - \eta_{\mathbf{i_2i_3}}\mathbf{i_1i_3} - \eta_{\mathbf{i_1i_3}}\mathbf{i_2i_3} - \eta_{\mathbf{i_3}}\mathbf{i_1i_2i_3}$;
3) $f(\eta) = \eta_1 + \eta_{\mathbf{i_1}}\mathbf{i_1} + \eta_{\mathbf{i_3}}\mathbf{i_2} + \eta_{\mathbf{i_1i_3}}\mathbf{i_1i_2} + \eta_{\mathbf{i_2}}\mathbf{i_3} + \eta_{\mathbf{i_1i_2}}\mathbf{i_1i_3} + \eta_{\mathbf{i_2i_3}}\mathbf{i_2i_3} + \eta_{\mathbf{i_1i_2i_3}}\mathbf{i_1i_2i_3}$;
4) $f(\eta) = \eta_1 + \eta_{\mathbf{i_1}}\mathbf{i_1} - \eta_{\mathbf{i_3}}\mathbf{i_2} - \eta_{\mathbf{i_1i_3}}\mathbf{i_1i_2} - \eta_{\mathbf{i_2}}\mathbf{i_3} - \eta_{\mathbf{i_1i_3}}\mathbf{i_1i_3} + \eta_{\mathbf{i_2i_3}}\mathbf{i_2i_3} + \eta_{\mathbf{i_1i_2i_3}}\mathbf{i_1i_2i_3}$.

4. A Bijection With Signed Permutations

The primarily goal of this section is to prove Theorem 1.2. We first need to show auxiliary results pertaining to multicomplex numbers.

4.1. **Auxiliary Results.** Recall that for any $\eta \in \mathbb{MC}(n)$, we can write
$$\eta = \sum_{\varepsilon \in \mathcal{E}_n} \eta_\varepsilon \boldsymbol{\varepsilon},$$
where $\eta_\varepsilon \in \mathbb{MC}(1)$. Write $\eta_\varepsilon = x_\varepsilon + \mathbf{i_1} y_\varepsilon$ with $x_\varepsilon, y_\varepsilon \in \mathbb{R}$. Therefore, for a real-linear ring homomorphism $f : \mathbb{MC}(n) \to \mathbb{MC}(n)$, we have
$$f(\eta) = \sum_{\varepsilon \in \mathcal{E}_n} x_\varepsilon f(\boldsymbol{\varepsilon}) + f(\mathbf{i_1}) \sum_{\varepsilon \in \mathcal{E}_n} y_\varepsilon f(\boldsymbol{\varepsilon}).$$

As an immediate consequence, we obtain the following proposition.

Proposition 4.1. *An involution of* $\mathbb{MC}(n)$ *is completely determined by its value on* $\mathbf{i_1}$ *and on the set* $\mathcal{E}_n$*.*

To choose the value $f(\mathbf{i_1})$, a key ingredient is the following observation.

Lemma 4.2. *Let* f *be an involution of* $\mathbb{MC}(n)$*. Then the following assertions hold.*

(i) Given a multicomplex number η *such that* $\eta^2 = -1$*, there exists a unique* $h \in H_n$ *such that* $\eta = \mathbf{i_1} h$*.*
(ii) $f(\mathbf{i_1}) = \mathbf{i_1} h$ *for a choice of* $h \in H_n$ *that depends on* f*.*

Proof. To prove (i), apply Proposition 2.1 with $u = \mathbf{i_1}$. Part (ii) follows from (i) and the fact that $f(\mathbf{i_1})^2 = -1$. □

Now we shall see how f acts on $\mathcal{E}_n$. Let η be an element of E_n. Define the set

$$\operatorname{orth}(\eta) = \{\zeta : \zeta^2 = \zeta,\, \eta\zeta = 0\},$$

that is, the set of idempotent elements orthogonal to η. We write

$$\eta = \sum_{\varepsilon \in \mathcal{E}_n} \eta_\varepsilon \boldsymbol{\varepsilon} \tag{4.1}$$

and $v(\eta)$ for the number of coefficients equal to 0 in the right-hand side of (4.1). If the number ζ is such that $\zeta^2 = \zeta$ and $\zeta = \sum_{\varepsilon \in \mathcal{E}_n} \zeta_\varepsilon \boldsymbol{\varepsilon}$, then the equality $\zeta\eta = 0$ is equivalent to $\eta_\varepsilon \zeta_\varepsilon = 0$ for all $\boldsymbol{\varepsilon} \in \mathcal{E}_n$. If $\eta_\varepsilon = 0$, ζ_ε can take the values 0 or 1, while if $\eta_\varepsilon = 1$, ζ_ε must be equal to 0. We thus have

1) $\#\operatorname{orth}(\eta) = 2^{v(\eta)}$;
2) $\#\operatorname{orth}(0) = 2^{2^{n-1}}$;
3) If $\boldsymbol{\varepsilon} \in \mathcal{E}_n$, then $\#\operatorname{orth}(\boldsymbol{\varepsilon}) = 2^{2^{n-1}-1}$;
4) If $\eta \neq 0$ and $\eta \notin \mathcal{E}_n$, then $\#\operatorname{orth}(\eta) < 2^{2^{n-1}-1}$.

In the above statements, the notation $\#A$ means the cardinality of the set A. A second key ingredient in proving our main result is the following lemma describing the action of f on the elements of $\mathcal{E}_n$.

Lemma 4.3. *Let* f *be a bijection from* $\mathbb{MC}(n)$ *to* $\mathbb{MC}(n)$ *such that* $f(0) = 0$ *and* $f(\eta\zeta) = f(\eta)f(\zeta)$ *for all* $\eta, \zeta \in \mathbb{MC}(n)$*. Let* η *be such that* $\eta^2 = \eta$*. Then*

(i) $(f(\eta))^2 = f(\eta)$*;*
(ii) $\#\operatorname{orth}(f(\eta)) = \#\operatorname{orth}(\eta)$*;*
(iii) If $\boldsymbol{\varepsilon} \in \mathcal{E}_n$*, then* $f(\boldsymbol{\varepsilon}) \in \mathcal{E}_n$*.*

Proof. The first part of the lemma follows directly from the fact that

$$f(\eta) = f(\eta^2) = f(\eta)f(\eta).$$

To prove part (ii), assume that $\zeta \in \operatorname{orth}(\eta)$. We then have

$$f(\zeta)f(\eta) = f(\zeta\eta) = f(0) = 0.$$

Since f is a bijection, the converse is also true, that is, if ζ is such that $f(\zeta) \in \operatorname{orth}(f(\eta))$, then $\zeta \in \operatorname{orth}(\eta)$. Therefore, $\zeta \in \operatorname{orth}(\eta)$ if and only if $f(\zeta) \in \operatorname{orth}(f(\eta))$. This, together with the fact that $f(0) = 0$, implies that

$$\#\operatorname{orth}(f(\eta)) = \#\operatorname{orth}(\eta).$$

Suppose now that $\boldsymbol{\varepsilon} \in \mathcal{E}_n$. Since $f(0) = 0$ and f is bijective, we have $f(\boldsymbol{\varepsilon}) \neq 0$. This implies

$$2^{2^{n-1}-1} = \#\operatorname{orth}(\boldsymbol{\varepsilon}) = \#\operatorname{orth}(f(\boldsymbol{\varepsilon})) = 2^{v(f(\boldsymbol{\varepsilon}))}$$

and thus $v(f(\boldsymbol{\varepsilon})) = 2^{n-1} - 1$. We deduce $f(\boldsymbol{\varepsilon}) \in \mathcal{E}_n$. □

Knowing how f acts on $\mathbf{i_1}$ and on $\mathcal{E}_n$, we can show the following.

Lemma 4.4. *Write $\boldsymbol{\varepsilon_j}$, $1 \leq j \leq 2^{n-1}$, for the elements of $\mathcal{E}_n$. Suppose that $f(\mathbf{i_1}) = \mathbf{i_1}h$ with*

$$h = \sum_{j=1}^{2^{n-1}} \eta_{\boldsymbol{\varepsilon_j}} \boldsymbol{\varepsilon_j}.$$

Suppose that $f(\boldsymbol{\varepsilon_j}) = \boldsymbol{\varepsilon_k}$. Then

$$f(\mathbf{i_1}\boldsymbol{\varepsilon_j}) = \mathbf{i_1}\eta_{\boldsymbol{\varepsilon_k}}\boldsymbol{\varepsilon_k}. \tag{4.2}$$

Furthermore, the function f is completely determined by its action on the set $\mathbf{i_1}\mathcal{E}_n$.

Proof. The formula (4.2) follows from direct computation and from the orthogonality of the elements of the set $\mathcal{E}_n$.

Suppose that we know the action of f on the set $\mathbf{i_1}\mathcal{E}_n$. Then, from the identity

$$f(\boldsymbol{\varepsilon_k}) = -\Big(f(\mathbf{i_1}\boldsymbol{\varepsilon_k})\Big)^2,$$

we can recover the value of $f(\boldsymbol{\varepsilon_k})$. The identity

$$f(\mathbf{i_1}) = \sum_{k=1}^{2^{n-1}} f(\mathbf{i_1}\boldsymbol{\varepsilon_k})$$

allows us to recover the value of $f(\mathbf{i_1})$. The result then follows from Proposition 4.1. □

4.2. **Proof of Theorem 1.2.** For a real number a, we define $\operatorname{sgn}(a) = 1$ if $a > 0$, $\operatorname{sgn}(0) = 0$ and $\operatorname{sgn}(a) = -1$ if $a < 0$. Also an alternative description of B_n, which is more useful to us, can be given as followed. Any signed permutation π can be seen as a bijection of $\{1, \ldots, n, -1, \ldots, -n\}$ to itself such that $\pi(-i) = -\pi(i)$ and $-(-i) = i$ for $i = 1, \ldots, n$. We have identified the bar with the $-$ sign in the previous description of a signed involution. The goal of this section is to prove the following result establishes a connection between real-linear automorphisms of $\mathbb{MC}(n)$ and $B_{2^{n-1}}$. We can then deduce a number of results from this connection.

Proof of Theorem 1.2. Let f be a real-linear automorphism of $\mathbb{MC}(n)$. From Lemma 4.4, f is determined by its action on the set $\mathbf{i_1}\mathcal{E}_n$ and $f(\mathbf{i_1}\boldsymbol{\varepsilon_j}) = \mathbf{i_1}\eta_{\boldsymbol{\varepsilon_k}}\boldsymbol{\varepsilon_k}$, with $\eta_{\boldsymbol{\varepsilon_k}} \in \{-1, 1\}$. To such a function f we can associate the signed permutation π that satisfies $\pi(j) = \eta_{\boldsymbol{\varepsilon_k}} k$. Conversely, for a given signed permutation π, we can define the function f by $f(\mathbf{i_1}\boldsymbol{\varepsilon_j}) = \mathbf{i_1}\operatorname{sgn}(\pi(j))\boldsymbol{\varepsilon}_{|\pi(j)|}$. It is clear from our construction that this bijection maps the identity to the identity and is compatible with composition. □

Using this bijection, we obtain the following corollary.

Corollary 4.5. *For each integer $n \geq 1$, there are $2^{2^{n-1}}(2^{n-1})!$ real-linear automorphisms of $\mathbb{MC}(n)$.*

Proof. For a given permutation of $\{1, \ldots, n\}$, we can define a signed permutation by choosing whether we put a bar or not over each entry. It is thus obvious that $\#B_n = 2^n \cdot n!$. The result follows. □

A more direct way to interpret the previous formula for the number of real-linear automorphisms of $\mathbb{MC}(n)$ corresponds to choosing a value for $f(\mathbf{i_1})$ and a permutation of $\mathcal{E}_n$. There are $(2^{n-1})!$ such permutations, and since $f(\mathbf{i_1})^2 = -1$, from Proposition 2.2 there are $2^{2^{n-1}}$ possible values for $f(\mathbf{i_1})$.

5. Characterization of Involutions of Multicomplex numbers

In this section, we derive the following result on the number of involutions of $\mathbb{MC}(n)$.

Corollary 5.1. *For $n \geq 1$ a positive integer, write $F(n)$ for the number of involutions of $\mathbb{MC}(n)$.*

(i) The following formula holds:

$$F(n) = (2^{n-1})! \sum_{k=0}^{\lfloor 2^{n-2} \rfloor} \frac{2^{2^{n-1}-2k}}{k!(2^{n-1}-2k)!}.$$

(ii) *If* $g(1) = 2$, $g(2) = 6$, *and*

$$g(n) = 2g(n-1) + (2n-2)g(n-2), \qquad n \geq 3,$$

then $F(n) = g(2^{n-1})$.

(iii) *The asymptotics*

$$F(n) \sim \left(\frac{2^n}{e}\right)^{2^{n-2}} \frac{e^{2^{n/2}}}{\sqrt{2e}}$$

hold as $n \to \infty$.

We first show how to obtain the above result using Theorem 1.2. We then present a more direct alternative approach to obtain the formula for the number of involutions of $\mathbb{MC}(n)$ in Corollary 5.1(i), using a counting argument. The main reason we choose to include this second proof is that this approach gives more insight into the nature of the multicomplex numbers since it relies on representation theorems for important subrings of $\mathbb{MC}(n)$.

5.1. **Proof of Corollary 5.1.** The formula in (i) and the asymptotic formula in (iii) follow from known results for signed involutions (see [5] and [24], respectively).

For (ii), by Theorem 1.2 it suffices to prove that $g(n)$ counts the number of signed involutions of length n. But $\pi \in B_n$ either fixes n and sends it to n or $-n$, or it sends n to j or $-j$, where $j \in \{1, 2, \ldots, n-1\}$. Establishing the base cases $g(1)$ and $g(2)$ is straightforward. □

5.2. **Counting involutions again.** Recall from Lemma 4.2 that $f(\mathbf{i_1}) = \mathbf{i_1}h$ for a unique $h \in H_n$ that depends on f. Assuming that f is an involution, the next result gives a way to choose h.

Lemma 5.2. *Let* f *be an involution of* $\mathbb{MC}(n)$ *for* $n \geq 1$. *Suppose that* $f(\mathbf{i_1}) = \mathbf{i_1}h$ *for some* $h \in H_n$. *Then, we have* $f(h) = h$.

Proof. Apply f on $f(\mathbf{i_1}) = \mathbf{i_1}h$ to get

$$\mathbf{i_1} = f(\mathbf{i_1})f(h).$$

Using again $f(\mathbf{i_1}) = \mathbf{i_1}h$, we see that

$$\mathbf{i_1} = \mathbf{i_1}hf(h). \tag{5.1}$$

Since $\mathbf{i_1}$ is invertible in $\mathbb{MC}(n)$, we obtain $1 = hf(h)$. Multiplying by h, we therefore obtain $h = f(h)$. □

Based on this last lemma, we introduce the sets

$$Y_n := \left\{ \sum_{h \in H_n} r_h h \,:\, r_h \in \mathbb{R} \right\} \quad \text{and} \quad \operatorname{fix}(f) := \{\eta \in Y_n \,:\, f(\eta) = \eta\}. \tag{5.2}$$

It is easy to see that Y_n is a vector subspace and a subring of $\mathbb{MC}(n)$ containing H_n and $\operatorname{fix}(f)$ is a vector subspace and a subring of Y_n if f is an involution. We can now prove the formula for $F(n)$ again.

Second proof of Corollary 5.1(i). Let f be an involution of $\mathbb{MC}(n)$. We still have that f is determined by its action on $\mathbf{i_1}$ and on $\mathcal{E}_n$, and that it induces a permutation of the elements of this set. For $n = 1$, we already know that there are 2 involutions on the complex space. This corresponds to what is computed with our formula.

For $n \geq 2$, since f is an involution, the permutation induced by f should contain cycles of length 2 (transpositions) and should fix some elements of $\mathcal{E}_n$. Let $k \in \{0, 1, \ldots, 2^{n-2}\}$. The number of permutations of a set of 2^{n-1} elements with k transpositions and $2^{n-1} - 2k$ fixed elements is given by

$$\frac{(2^{n-1})!}{2^k k! (2^{n-1} - 2k)!}.$$

We now need to find the possible values of $f(\mathbf{i_1})$. We know that

$$f(\mathbf{i_1}) = \mathbf{i_1} h$$

for some choice of $h \in \operatorname{fix}(f)$ by Lemma 4.2 and Lemma 5.2. Since $\operatorname{fix}(f)$ is a subring of Y_n and f is real-linear, it is sufficient to know the coefficients of h in its representation with respect to a basis of $\operatorname{fix}(f)$. We will find such a basis. Denote the elements of $\mathcal{E}_n$ by $\boldsymbol{\varepsilon_1}, \ldots, \boldsymbol{\varepsilon_{2^{n-1}}}$. Suppose that f is given by

$$f(\boldsymbol{\varepsilon_{j_1}}) = \boldsymbol{\varepsilon_{j_2}},\ f(\boldsymbol{\varepsilon_{j_3}}) = \boldsymbol{\varepsilon_{j_4}},\ \ldots,\ f(\boldsymbol{\varepsilon_{j_{2k-1}}}) = \boldsymbol{\varepsilon_{2k}}$$

and $f(\boldsymbol{\varepsilon_{j_{2k+1}}}) = \boldsymbol{\varepsilon_{j_{2k+1}}}, \ldots, f(\boldsymbol{\varepsilon_{j_{2^{n-1}}}}) = \boldsymbol{\varepsilon_{j_{2^{n-1}}}}$. From Corollary 2.4(ii), we can write

$$h = \sum_{\ell=1}^{2^{n-1}} c_\ell \boldsymbol{\varepsilon_{j_\ell}}$$

with $c_\ell \in \{-1, 1\}$. We then have

$$f(h) = \sum_{\ell=1}^{2k-1} (c_{\ell+1} \boldsymbol{\varepsilon_{j_\ell}} + c_\ell \boldsymbol{\varepsilon_{j_{\ell+1}}}) + \sum_{\ell=2k+1}^{2^{n-1}} c_\ell \boldsymbol{\varepsilon_{j_\ell}}.$$

It follows that $f(h) = h$ if and only if $c_1 = c_2, c_3 = c_4, \ldots, c_{2k-1} = c_{2k}$. This is equivalent to stating that

$$\mathrm{fix}(f) = \mathrm{span}_{\mathbb{R}}\{(\boldsymbol{\varepsilon_{j_1}} + \boldsymbol{\varepsilon_{j_2}}),\ \ldots,\ (\boldsymbol{\varepsilon_{j_{2k-1}}} + \boldsymbol{\varepsilon_{j_{2k}}}), \boldsymbol{\varepsilon_{j_{2k+1}}},\ \ldots,\ \boldsymbol{\varepsilon_{j_{2^{n-1}}}}\}.$$

We deduce that h should be of the form

$$h = \sum_{\substack{\ell=1 \\ \ell \text{ odd}}}^{2k-1} a_\ell(\boldsymbol{\varepsilon_{j_\ell}} + \boldsymbol{\varepsilon_{j_{\ell+1}}}) + \sum_{\ell=2\ell+1}^{2^{n-1}} a_\ell \boldsymbol{\varepsilon_{j_\ell}},$$

with $a_\ell \in \{-1, 1\}$. This implies that the number of ways to choose h is

$$2^{2^{n-1}-k}.$$

Finally, summing from $k = 0$ to $k = 2^{n-2}$, we obtain that the number of involutions of $\mathbb{MC}(n)$ for $n \geq 2$ is

$$\sum_{k=0}^{2^{n-2}} \frac{2^{2^{n-1}-k}(2^{n-1})!}{2^k k!(2^{n-1} - 2k)!} = (2^{n-1})! \sum_{k=0}^{2^{n-2}} \frac{2^{2^{n-1}-2k}}{k!(2^{n-1} - 2k)!}.$$

This completes the proof. □

6. Higher Order Involutions

In the previous section, we investigated involutions, functions f such that $f^{(2)} = \mathrm{Id}$. It is natural to ask what happens when 2 is replaced by an arbitrary positive integer $r \geq 2$.

We write $F_r(n)$ for the number of real-linear automorphisms of $\mathbb{MC}(n)$ such that $f^{(r)} = \mathrm{Id}$. From Theorem 1.2, the set of real-linear automorphisms f of $\mathbb{MC}(n)$ such that $f^{(r)} = \mathrm{Id}$ can be identified with the set of signed permutations $\pi \in B_{2^{n-1}}$ such that $\pi^{(r)} = \mathrm{Id}$. We will thus enumerate such signed permutations in order to prove our next result.

Theorem 6.1. *Let $p > 2$ be a prime number. The number of p-involutions of the multicomplex numbers $\mathbb{MC}(n)$ with $n \geq 1$ is given by*

$$F_p(n) = (2^{n-1})! \sum_{k=0}^{\lfloor 2^{n-1}/p \rfloor} \frac{2^{(p-1)k}}{k! p^k (2^{n-1} - pk)!}.$$

More generally, if $r > 1$ is a positive integer, the number of r-involutions of the multicomplex numbers $\mathbb{MC}(n)$ with $n \geq 1$ is given by

$$F_r(n) = 2^{2^{n-1}} \sum_{\sigma \in S_{2^{n-1},r}} \Bigg(\prod_{k|r,\, r/k \,\text{is odd}} \frac{1}{2^{\mathrm{cyc}_k(\sigma)}} \Bigg).$$

Proof. For a signed permutation π, we let $\sigma = \mathrm{un}(\pi)$ stand for the corresponding unsigned permutation. Clearly, $\pi^{(r)} = \mathrm{Id}$ implies $\sigma^{(r)} = \mathrm{Id}$ while the converse is not true. Recall that $S_{n,t}$ stands for the set of permutations σ of n elements such that $\sigma^{(t)} = \mathrm{Id}$. We have directly

$$F_r(n) = \sum_{\sigma \in S_{2^{n-1},r}} \# \left\{ \pi \in B_{2^{n-1}} : \mathrm{un}(\pi) = \sigma, \pi^{(r)} = \mathrm{Id} \right\}.$$

The cardinalities of the sets in the above sum correspond to the number of ways of choosing the signs of the signed permutations. Any unsigned permutation σ can be written as a product of disjoint cycles. Under the assumption that $\sigma^{(r)} = \mathrm{Id}$, we have that the lengths of these cycles are divisors of r. Assume that σ has a cycle of length s and assume without loss of generality that this cycle is $(1, 2, \ldots, s)$. Let $c_1, c_2, \ldots, c_s$ be the signs associated to the elements $1, 2, \ldots, s$ in the signed permutation π. We have

$$\pi^{(r)}(j) = (c_1 \cdots c_s)^{r/s} j, \qquad 1 \leq j \leq s,$$

and thus

$$\pi^{(r)}(j) = j \text{ for } 1 \leq j \leq s \quad \iff \quad (c_1 \cdots c_s)^{r/s} = 1.$$

If r/s is even, then the signs $c_1, \ldots, c_s$ can be chosen arbitrarily and the number of possible choices is equal to 2^s. On the other hand, if r/s is odd then $c_1 \cdots c_s$ must be equal to 1. The number of possible choices in this case is thus 2^{s-1}. For a given unsigned permutation σ, we denote by $\mathrm{cyc}_s(\sigma)$ the number of disjoint cycles of length s in σ. Hence, the cardinality of the set $\left\{ \pi \in B_{2^{n-1}} : \mathrm{un}(\pi) = \sigma, \pi^{(r)} = \mathrm{Id} \right\}$ is

$$\prod_{s|r, r/s \text{ is even}} 2^{s \cdot \mathrm{cyc}_s(\sigma)} \prod_{s|r, r/s \text{ is odd}} 2^{(s-1)\mathrm{cyc}_s(\sigma)}. \tag{6.1}$$

By noticing that for a fixed permutation σ, $\sum_{s|r} s \cdot \mathrm{cyc}_s(\sigma) = 2^{n-1}$, the last expression can be rewritten as

$$2^{2^{n-1}} \prod_{s|r, r/s \text{ is odd}} \frac{1}{2^{\mathrm{cyc}_s(\sigma)}}. \tag{6.2}$$

Summing over all permutations $\sigma \in S_{2^{n-1},r}$, we get

$$F_r(n) = 2^{2^{n-1}} \sum_{\sigma \in S_{2^{n-1},r}} \prod_{s|r, r/s \text{ is odd}} \frac{1}{2^{\mathrm{cyc}_s(\sigma)}}.$$

When $r = p$ is an odd prime, expression (6.1) simplifies to

$$2^{(1-1)\mathrm{cyc}_1(\sigma)} \cdot 2^{(p-1)\mathrm{cyc}_p(\sigma)} = 2^{(p-1)\mathrm{cyc}_p(\sigma)}. \tag{6.3}$$

The number of permutations of 2^{n-1} elements with k cycles of length p and $2^{n-1} - pk$ fixed elements is given by

$$\frac{(2^{n-1})!}{p^k k!(2^{n-1} - pk)!}. \tag{6.4}$$

From (6.3) and (6.4), we conclude that for an odd prime p,

$$F_p(n) = (2^{n-1})! \sum_{k=0}^{\lfloor 2^{n-1}/p \rfloor} \frac{2^{k(p-1)}}{p^k k!(2^{n-1} - pk)!},$$

and this concludes the proof. □

Note that the above argument does not work when $r = 2$, although the expression (6.4) still holds. In this case, (6.2) simplifies to

$$2^{2^{n-1}} \frac{1}{2^{\mathrm{cyc}_2(\sigma)}} = 2^{2^{n-1}-k}, \tag{6.5}$$

where k is the number of cycles of length 2 in the unsigned permutation σ. From (6.4) and (6.5), we therefore obtain

$$F(n) = F_2(n) = (2^{n-1})! \sum_{k=0}^{2^{n-2}} \frac{2^{2^{n-1}-k}}{k! 2^k (2^{n-1} - 2k)!} = (2^{n-1})! \sum_{k=0}^{2^{n-2}} \frac{2^{2^{n-1}-2k}}{k!(2^{n-1} - 2k)!},$$

as before.

6.1. **Generating r-Involutions.** The proof of Corollary 6.1 combined with the explicit bijection in the proof of Theorem 1.2 gives a way to generate the r-involutions of $\mathbb{MC}(n)$, for $r \geq 2$. Here, we describe this method (which is a brute force method).

1) Select a permutation $\sigma \in S_{2^{n-1},r}$ of the symbols $\{1, 2, \ldots, 2^{n-1}\}$.
2) Generate all the possible sign permutations π by considering all the possible sign insertions in σ.
3) For a given sign permutation in the last step, check if $\pi^{(r)} = \mathrm{Id}$.
4) Generate the r-involution of $\mathbb{MC}(n)$ by setting

$$f(\mathbf{i_1} \boldsymbol{\varepsilon_j}) = \mathbf{i_1} \mathrm{sgn}(\pi(j)) \boldsymbol{\varepsilon}_{|\boldsymbol{\pi(j)}|}.$$

Using this method we can generate an example of a 6-involution of the space $\mathbb{MC}(3)$ that do not preserve the set $\mathbb{I}(3)$.

Let $\boldsymbol{\varepsilon_1} = (1 - \mathbf{i_1 i_2} - \mathbf{i_1 i_3} - \mathbf{i_2 i_3})/4$, $\boldsymbol{\varepsilon_2} = (1 + \mathbf{i_1 i_2} + \mathbf{i_1 i_3} - \mathbf{i_2 i_3})/4$, $\boldsymbol{\varepsilon_3} = (1 - \mathbf{i_1 i_2} + \mathbf{i_1 i_3} + \mathbf{i_2 i_3})/4$, and $\boldsymbol{\varepsilon_4} = (1 + \mathbf{i_1 i_2} - \mathbf{i_1 i_3} + \mathbf{i_2 i_3})/4$ and define the action of f as followed:

$$f(\mathbf{i_1}\boldsymbol{\varepsilon_1}) = \mathbf{i_1}\boldsymbol{\varepsilon_3},\ f(\mathbf{i_1}\boldsymbol{\varepsilon_2}) = \mathbf{i_1}(-1)\boldsymbol{\varepsilon_2},\ f(\mathbf{i_1}\boldsymbol{\varepsilon_3}) = \mathbf{i_1}\boldsymbol{\varepsilon_4},\ f(\mathbf{i_1}\boldsymbol{\varepsilon_4}) = \mathbf{i_1}\boldsymbol{\varepsilon_1}.$$

This map is a 6-involution. It comes from the following signed permutation (using the bar notation):

$$\pi = 3\overline{2}41.$$

Note that we could create a 3-involution by using the unsigned permutation

$$\sigma = 3241.$$

We can rewrite the above 6-involution f using the elementary units (canonical representation) as follows:

$$\begin{aligned} f(\eta) = \eta_1 &+ \mathbf{i_1}(\eta_{\mathbf{i_1 i_2 i_3}} + \eta_{\mathbf{i_1}} + \eta_{\mathbf{i_2}} + \eta_{\mathbf{i_3}})/2 + \mathbf{i_2}(-\eta_{\mathbf{i_1 i_2 i_3}} + \eta_{\mathbf{i_1}} - \eta_{\mathbf{i_2}} + \eta_{\mathbf{i_3}})/2 \\ &+ \mathbf{i_1 i_2}\eta_{\mathbf{i_1 i_3}} + \mathbf{i_3}(\eta_{\mathbf{i_1 i_2 i_3}} + \eta_{\mathbf{i_1}} - \eta_{\mathbf{i_2}} - \eta_{\mathbf{i_3}})/2 - \mathbf{i_1 i_3}\eta_{\mathbf{i_2 i_3}} - \mathbf{i_2 i_3}\eta_{\mathbf{i_1 i_2}} \\ &+ \mathbf{i_1 i_2 i_3}(-\eta_{\mathbf{i_1 i_2 i_3}} + \eta_{\mathbf{i_1}} + \eta_{\mathbf{i_2}} - \eta_{\mathbf{i_3}})/2, \end{aligned}$$

where

$$\eta = \eta_1 + \mathbf{i_1}\eta_{\mathbf{i_1}} + \mathbf{i_2}\eta_{\mathbf{i_2}} + \mathbf{i_1 i_2}\eta_{\mathbf{i_1 i_2}} + \mathbf{i_3}\eta_{\mathbf{i_3}} + \mathbf{i_1 i_3}\eta_{\mathbf{i_1 i_3}} + \mathbf{i_2 i_3}\eta_{\mathbf{i_2 i_3}} + \mathbf{i_1 i_2 i_3}\eta_{\mathbf{i_1 i_2 i_3}}.$$

This 6-involution is not $\mathbb{I}(3)$-preserving because it sends some of the elementary units in $\mathbb{I}(3)$ to units in $U_3 \backslash \mathbb{I}(3)$. From the expression above, we see that

$$f(\mathbf{i_1}) = \frac{\mathbf{i_1} + \mathbf{i_2} + \mathbf{i_3} + \mathbf{i_1 i_2 i_3}}{2}, \; f(\mathbf{i_2}) = \frac{\mathbf{i_1} - \mathbf{i_2} - \mathbf{i_3} + \mathbf{i_1 i_2 i_3}}{2},$$

and

$$f(\mathbf{i_3}) = \frac{\mathbf{i_1} + \mathbf{i_2} - \mathbf{i_3} - \mathbf{i_1 i_2 i_3}}{2}.$$

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